Text

INTERMEDIATE

ALGEBRA

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INTERMEDIATE ALGEBRA

CHAPTER I

Quadratic Equations With One Variable

1·1. An equation is said to be in rational and integral form if the unknown quantity x, involved in the equation, has only positive integral powers. An equation may be rational and integral even if the co-efficients of different terms are fractional or irrational. Thus the equation $\frac{4}{9}x^2+x\sqrt{3}+2=0$, is in rational and integral form in spite of the fact that some of the co-efficients are fractional or irrational. On the other hand, the equation $x+\frac{6}{x^2}=5$, is not in the rational and integral form, for the second term involving x is raised to a negative power -2.

An equation in the rational and integral form may be either linear, quadratic, cubic or biquadratic according as the highest degree term in x is x, x^2, x^3 or x^4. Generally, if x be raised to the power n and to no higher power, the equation is said to be of degree n.

A number h is said to be a **root** of an equation, if the equation is satisfied, when x is replaced by h. The equation is said to be completely solved, if all its roots are known.

In this chapter we shall deal with quadratic equations and with those of a higher degree which are capable of reduction to the quadratic form.

The most general form of the quadratic equation is $ax^2+bx+c=0$, where a, b, c are constants, positive or negative, rational or irrational. Here

a is the co-efficient of x^2

b is the co-officient of x

and c is the absolute term.

In the equation $2x^2+3x-7=0$, $a=2$, $b=3$, $c=-7$.

Similarly in the equation $\sqrt{2}-5x^2=0$, $a=-5$, $b=0$, $c=\sqrt{2}$.

1·2. Solution of Quadratic Equations. It is easy to solve a quadratic equation in which the co-efficient of x is zero. For, if it be required to solve the equation $ax^2+c=0$, we write $x^2=\frac{-c}{a}$ so that

$$x=\sqrt{\frac{-c}{a}} \text{ or } -\sqrt{\frac{-c}{a}} \text{ at once.}$$

Remember that every positive number has two square roots, which are equal in magnitude but opposite in sign.

In general if the solution of a quadratic equation be required there are two methods by which we may proceed. The first, known as the method of factors, is illustrated by the following examples. The second is to obtain and remember a formula for the solution of the general quadratic equation $ax^2+bx+c=0$ and use it wherever required.

Ex. Solve by method of factors the following equations :

(*i*) $x^2-5x+6=0$, (*ii*) $4x^2-4ax+(a^2-b^2)=0$.

Sol. (*i*) The equation may be written as

$$(x-2)(x-3)=0$$

The left hand side will be zero when

$$x-2=0 \text{ i.e., when } x=2.$$

It will also be zero when $x-3=0$, *i.e.*, when $x=3$.
Thus we find that 2, 3 are the roots of the given equation.

(*ii*) The given equation may be written as

$$4x^2-2(a+b)x-2(a-b)x+(a^2-b^2)=0$$

i.e., $\quad 2x[2x-(a+b)]-(a-b)[2x-(a+b)]=0$

i.e., $\quad [2x-(a+b)][2x-(a-b)]=0$

$\therefore$ either $2x-(a+b)=0$ or $2x-(a-b)=0$

Hence $x=\frac{a+b}{2}, \frac{a-b}{2}$ are the roots.

1·3. Roots of the General Quadratic. In order to solve the equation $ax^2+bx+c=0$, we transpose c to the other side and write

$$ax^2+bx=-c \qquad \text{(First step)}$$

Dividing throughout by a,

we get $$x^2+\frac{b}{a}x=\frac{-c}{a}$$ (Second step)

In order to complete the square on the left-hand side, we add $\frac{b^2}{4a^2}$ to both the sides, getting

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$ (Third step)

i.e., $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

whence $$x+\frac{b}{2a}=\frac{\sqrt{b^2-4ac}}{2a} \text{ or } -\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ or } \frac{-b-\sqrt{b^2-4ac}}{2a}$$

or we may write $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Note. We have seen that a quadratic equation has two roots. It will be proved at a later stage that it cannot have more than two roots.

Ex. Solve by using the formula the following equations :
(*i*) $x^2-7x+12=0$; (*ii*) $pqx^2-(p^2+q^2)x+pq=0$.

Sol. (*i*) Here $a=1,\ b=-7,\ c=12$,

$$\therefore \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-7)\pm\sqrt{(-7)^2-4\times1\times12}}{2\times1}$$

$$=\frac{7\pm\sqrt{49-48}}{2}=\frac{7\pm1}{2}=4 \text{ or } 3$$

(*ii*) Here $a=pq,\ b=-(p^2+q^2),\ c=pq$.

$\therefore$ As before

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{p^2+q^2\pm\sqrt{(p^2+q^2)^2-4p^2q^2}}{2pq}$$

$$=\frac{(p^2+q^2)\pm(p^2-q^2)}{2pq}=\frac{p}{q} \text{ or } \frac{q}{p}.$$

1·4. If an equation is not in the rational and integral form, we must first reduce it to that form and then proceed to solve

it. The process of reduction sometimes becomes laborious, unless special devices are employed. The following examples are instructive in this connection.

(*i*) $x+\frac{1}{x}=3\frac{1}{3}$ (*P. U.* 1934)

Multiplying throughout by $3x$, and transposing we get $3x^2-10x+3=0$, which may now be solved. The roots are 3 and $\frac{1}{3}$.

(*ii*) $x=\cfrac{a}{b+\cfrac{c}{d+x}}$

Simplifying the continued fraction on the right, we get

$$x=\frac{a(d+x)}{b(d+x)+c}$$

$\therefore \quad (bd+c)x+bx^2=ad+ax$

or $bx^2+(c-a+bd)x-ad=0$, which can now be solved by using the general formula.

(*ii*) $\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}=\frac{3}{x}$.

Here in stead of adding together all the fractions on the left-hand side, it would be better to write $\frac{1}{x}+\frac{1}{x}+\frac{1}{x}$ in place of $\frac{3}{x}$ and regroup the terms suitably. Thus the equation may be written as

$$\left(\frac{1}{x+1}-\frac{1}{x}\right)+\left(\frac{1}{x+2}-\frac{1}{x}\right)+\left(\frac{1}{x+3}-\frac{1}{x}\right)=0$$

whence $-\frac{1}{x(x+1)}-\frac{2}{x(x+2)}-\frac{3}{x(x+2)}=0.$

$\therefore$ either $\frac{1}{x}=0$ or $\frac{1}{x+1}+\frac{2}{x+2}+\frac{3}{x+3}=0$

The first gives infinite value of x, and the second gives

$$(x+2)(x+3)+2(x+3)(x+1)+3(x+1)(x+2)=0$$

i.e., $6x^2+22x+18=0$, which can now be solved.

(*iv*) $\frac{(x-a)(x-b)}{x-a-b}=\frac{(x-c)(x-d)}{x-c-d}$ (Math. Trip.)

The equation may be written as

$$\frac{x(x-a-b)+ab}{x-a-b}=\frac{x(x-c-d)+cd}{x-c-d}.$$

i. e., $$x+\frac{ab}{x-a-b}=x+\frac{cd}{x-c-d}$$

i. e., $$ab(x-c-d)=cd(x-a-b)$$

whence $$x=\frac{ab(c+d)-cd(a+b)}{ab-cd}.$$

(v) $$\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}.$$

The equation may be written as

$$\frac{1}{a+b+x}-\frac{1}{x}=\frac{1}{a}+\frac{1}{b}$$

i. e., $$\frac{x-(a+b+x)}{x(a+b+x)}=\frac{a+b}{ab}$$

i. e., $\dfrac{1}{ab}+\dfrac{1}{x(a+b+x)}=0$; *i. e.,* $x^2+(a+b)x+ab=0$

which can now be solved.

Exercises.

Solve the following equations :—

1. $x^2+4x-60=0$.
2. $x^2-(a+b)x+ab=0$.
3. $15+4x-4x^2=0$.
4. $(x+3)(x+2)=2$.
5. $\dfrac{x}{2}-\dfrac{2}{x}=\dfrac{3}{2}$.
6. $\dfrac{1}{x-a}+\dfrac{1}{x-b}+\dfrac{1}{x-c}=0$. (*P.U*).
7. $\dfrac{2x(a-x)}{3a-2x}=\dfrac{a}{4}$. (*P. U.*)
8. $9(x-2)-4(x-3)=x^2$
9. $x^2-3\sqrt{2}x+4=0$.
10. $x^2-\sqrt{2}(\sqrt{2}+1)x+(\sqrt{2}+1)=0$.
11. $\dfrac{3}{5-x}+\dfrac{2}{4-x}=\dfrac{8}{x+2}$. (*P. U.*)
12. $\dfrac{1}{x-1}+\dfrac{2}{x-2}+\dfrac{3}{x-3}=0$. (*P. U.*)

13. $\frac{3+2x}{1+2x}-\frac{5+2x}{7+2x}=1-\frac{4x^2-2}{7+6x+4x^2}$. (P. U.)

14. $\frac{x}{x+1}+\frac{x+1}{x+2}+\frac{x+2}{x+3}=3$. (P. U.)

15. $\frac{a}{x+a}+\frac{b}{x+b}=\frac{a-c}{x+a-c}+\frac{b+c}{x+b+c}$. (P. U.)

16. $\frac{a}{ax-1}+\frac{b}{bx-1}=a+b$. (P. U.)

17. $\frac{x-7}{x-9}-\frac{x-9}{x-11}=\frac{x-13}{x-15}-\frac{x-15}{x-17}$. (P. U.)

18. $\frac{4}{x+4}-\frac{3}{x+3}=\frac{2}{x+2}-\frac{1}{x+1}$. (P. U.)

19. $\frac{1}{2x+a+b}+\frac{1}{2x-a+b}+\frac{1}{2x+a-b}+\frac{1}{2x-a-b}=0$.

1·5. **Reduction of irrational equations.** The following examples illustrate important devices to be employed when the given equation involves square or cube roots of quadratic or linear expressions in the unknown quantity.

(*i*) $\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}+\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=62.$

Rationalising both the denominators on the left, we write

$$\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\times\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}+\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\times\frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}}=62$$

i. e., $\frac{(x+\sqrt{x^2-1})^2+(x-\sqrt{x^2-1})^2}{x^2-(x^2-1)}=62$

i. e., $2x^2+2(x^2-1)=62$ *i. e.,* $x^2=16$

$\therefore$ $x=\pm 4.$

(*ii*) $\sqrt{x}+\sqrt{x-3}=\sqrt{x+5}$

Squaring both sides, we have

$$x+(x-3)+2\sqrt{x(x-3)}=x+5$$

i. e., $(x-8)=-2\sqrt{x(x-3)}$

Squaring again $(x-8)^2=4x(x-3)$

i. e., $x^2-16x+64=4x^2-12x$

or $3x^2+4x-64=0.$

The roots of the last equation are found out to be 4, and $-\frac{16}{3}$. Out of these we find that 4 actually satisfies the given equation, whereas $-\frac{16}{3}$ does not. So by definition $-\frac{16}{3}$ is no a root of the given equation. Such a root is called an **extraneous root.**

It is important to note the stage at which extraneous roots are introduced in the solution of an equation. Suppose we square both sides of the equation $x=2$. We get $x^2=4$, whose roots are 2 and -2. But -2 is easily seen to be an extraneous root. It is not a root of the original equation but of the equation $x=-2$.

In the above example $\frac{-16}{3}$ is a root not of the original equation but of the equation $-\sqrt{x}+\sqrt{x-3}=\sqrt{x+5}$.

(*iii*) $\frac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}}=\frac{\sqrt{a}+1}{\sqrt{a}-1}$

By componendo and dividendo

$$\frac{(x+\sqrt{12a-x})+(x-\sqrt{12a-x})}{(x+\sqrt{12a-x})-(x-\sqrt{12a-x})}=\frac{(\sqrt{a}+1)+(\sqrt{a}-1)}{(\sqrt{a}+1)-(\sqrt{a}-1)}$$

i. e., $\frac{x}{\sqrt{12a-x}}=\sqrt{a}$ or $x^2=a(12a-x)$

i. e., $x^2+ax-12a^2=0$, whose roots are $3a$, $-4a$.

It is found that $-4a$ is an extraneous root of the given equation and is a root of the equation

$$\frac{x-\sqrt{12a-x}}{x+\sqrt{12a-x}}=\frac{\sqrt{a}+1}{\sqrt{a}-1}$$

(*iv*) $\frac{\sqrt{2+x}}{\sqrt{2}+\sqrt{2+x}}=\frac{\sqrt{2-x}}{\sqrt{2}-\sqrt{2-x}}$

On rationalising both the denominators in the equation. we have

$$\frac{\sqrt{2+x}\,(\sqrt{2}-\sqrt{2+x})}{2-(2+x)}=\frac{\sqrt{2-x}\,(\sqrt{2}+\sqrt{2-x})}{2-(2-x)}$$

$$\therefore \qquad \frac{\sqrt{4+2x}-(2+x)}{-x}=\frac{\sqrt{4-2x}+(2-x)}{x}$$

i. e., $\quad -(\sqrt{4+2x}-2-x)=\sqrt{4-2x}+2-x$

i. e., $\quad \sqrt{4+2x}+\sqrt{4-2x}=2x$

Squaring both sides

$$(4+2x)+(4-2x)+2\sqrt{16-4x^2}=4x^2$$

i. e., $\quad \sqrt{4-x^2}=x^2-2$

Again squaring both sides, $4-x^2=x^4-4x^2+4$

i. e., $\quad x^2(x^2-3)=0$ giving $x=0, \pm\sqrt{3}$.

By actual substitution we find that none of these three roots satisfies the equation.

(*v*) $\sqrt{x^2+x-2}-\sqrt{x^2-4x+3}=\sqrt{x^2-7x+6}$.

The equation may be written as

$$\sqrt{(x-1)(x+2)}-\sqrt{(x-1)(x-3)}=\sqrt{(x-1)(x-6)}$$

We find that $\sqrt{x-1}$ is common throughout

$\therefore$ either $\sqrt{x-1}=0$ giving $x=1$;

or, $\quad \sqrt{x+2}-\sqrt{x-3}=\sqrt{x-6}$

Squaring both sides we have

$$(x+2)+(x-3)-2\sqrt{(x+2)(x-3)}=x-6$$

$$\therefore \qquad (x+5)=2\sqrt{(x+2)(x-3)}.$$

$$\therefore \qquad (x+5)^2=4(x+2)(x-3)$$

i. e., $\quad 3x^2-14x-49=0$, giving $x=7, \frac{-7}{3}$.

$\therefore$ the roots are $1, 7, \frac{-7}{3}$.

Out of these $\frac{-7}{3}$ is an extraneous root. It is actually a root of the equation

$$\sqrt{x+2}+\sqrt{x-3}=\sqrt{x-6}.$$

Equations containing square roots of quadratic or linear expressions can sometimes be easily solved by employing an artifice illustrated by the following example.

(*vi*) $\sqrt{2x^2+3x-10}-\sqrt{x^2+3x-1}=x-3$..................(*i*)

We have identically (*i.e.* for all values of x)

$(2x^2+3x-10)-(x^2+3x-1)=x^2-9$..............(*ii*)

Dividing both sides of (*ii*) by the corresponding sides of (*i*), we have

$\sqrt{2x^2+3x-10}+\sqrt{x^2+3x-1}=x+3$..................(*iii*)

From (*i*) and (*iii*) by addition,

$$2\sqrt{2x^2+3x-10}=2x \text{ i.e., } \sqrt{2x^2+3x-10}=x.$$

$$\therefore 2x^2+3x-10=x^2 \text{ i.e., } x^2+3x-10=0,$$

whose roots are 2, -5.

It is found that -5 is an extraneous root and belongs to the equation

$$-\sqrt{2x^2+3x-10}-\sqrt{x^2+3x-1}=x-3.$$

(*vii*) $(a+x)^{\frac{1}{3}}+(b-x)^{\frac{1}{3}}=(a+b)^{\frac{1}{3}}$

Cubing both sides, we have

$$(a+x)+(b-x)+3(a+x)^{\frac{1}{3}}(b-x)^{\frac{1}{3}}[(a+x)^{\frac{1}{3}}+(b-x)^{\frac{1}{3}}]=a+b.$$

$$[\because (x+y)^3=x^3+y^3+3xy(x+y)]$$

But $(a+x)^{\frac{1}{3}}+(b-x)^{\frac{1}{3}}=(a+b)^{\frac{1}{3}}$,

$\therefore$ making this substitution and simplifying,

$$3(a+x)^{\frac{1}{3}}(b-x)^{\frac{1}{3}}(a+b)^{\frac{1}{3}}=0$$

i.e., $(a+x)^{\frac{1}{3}}(b-x)^{\frac{1}{3}}=0$. *i.e.*, $(a+x)(b-x)=0$,

$\therefore x=-a$ and b.

Exercises

Solve the equations :—

1. $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}=0.$ 2. $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}=3.$

3. $\dfrac{1+x}{1+\sqrt{1+x}}+\dfrac{1-x}{1+\sqrt{1-x}}=1.$

4. $\dfrac{1}{\sqrt{1-x}+1}+\dfrac{1}{\sqrt{1+x}-1}=\dfrac{1}{x}$

5. $\sqrt{a^2x^2-b^2}+(ax-b)=\sqrt{a^2x^2-3abx+2b^2}$.

6. $\sqrt{x^2-2x}+x=0$. 7. $\sqrt{x+2}+\sqrt{x+3}=\sqrt{2x+5}$.

8. $\sqrt{x^2-5x+6}+\sqrt{x^2-3x+2}=\sqrt{x^2-7x+10}$.

9. $\sqrt{x^2-x-2}+\sqrt{x^2+5x+4}=\sqrt{x+1}$.

10. $\dfrac{x-5}{\sqrt{x-1}+2}+\dfrac{x-7}{\sqrt{x-3}+2}=3$.

11. $\dfrac{2x-3}{\sqrt{x-2}+1}=2\sqrt{x-2}-1$.

12. $\sqrt{3x-2}+\sqrt{4x-3}=\sqrt{5x-4}+\sqrt{6x-5}$.

13. $\sqrt{x+a}+\sqrt{x+b}+\sqrt{x+c}=0$.

14. $\sqrt{ax+b}+\sqrt{cx+d}+\sqrt{ex+f}=0$.

15. $\sqrt{x^2+2x+4}+\sqrt{x^2+2x+9}=5$.

16. $\sqrt{2x^2+3x-4}-\sqrt{x^2+3x+5}=x-3$.

17. $\sqrt{2x^2+ax-6a^2}+\sqrt{x^2+ax-5a^2}=x+a$.

18. $\sqrt{x^2-3x+36}-\sqrt{x^2-3x+9}=3$. (*P. U.* 1941)

19. $\sqrt{1-x+x^2}-\sqrt{1+x-x^2}=x$.

20. $\sqrt{5x^2+7x+2}-\sqrt{4x^2+7x+18}=x-4$.

21. $\sqrt{ax^2+bx+c}+\sqrt{ax^2+bx+d}=\sqrt{c}+\sqrt{d}$.

1·6. Equations reducible to the quadratic by substitution.

Sometimes it happens that although the equation which is required to be solved is not quadratic, yet by effecting a proper substitution, we can arrive at a quadratic equation involving a new variable. The values of the new variable can thus be found out and since the new variable is connected with the old one by a specific relation, we can find the roots of the original equation. The following examples illustrate the different points that arise in this connection.

(*i*) $x^4-13x^2+36=0$.

Putting $x^2=y$, this equation becomes $z^2-13y+36=0$.

We get $y=4$ or 9, *i.e.*, $x^2=4$ or 9.

Now $x^2=4$, gives $x=\pm 2$
and $x^2=9$, ,, $x=\pm 3$.

$\therefore$ $\pm 2, \pm 3$ are the roots of the given equation.

(*ii*) $\sqrt{\dfrac{x^2+3}{x^2-3}}+8\sqrt{\dfrac{x^2-3}{x^2+3}}=6.$

Putting $\sqrt{\dfrac{x^2+3}{x^2-3}}=y$, we have $y+\dfrac{8}{y}=6$

$\therefore y^2-6y+8=0$, whence $y=2$ or 4

If $y=\sqrt{\dfrac{x^2+3}{x^2-3}}=2$, we have $\dfrac{x^2+3}{x^2-3}=4$

By componendo and dividendo

$$\frac{x^2}{3}=\frac{5}{3}, \text{ i.e., } x=\pm\sqrt{5}.$$

Similarly, if $y=\sqrt{\dfrac{x^2+3}{x^2-3}}=4$, we have $x=\pm\sqrt{\dfrac{17}{5}}$.

(*iii*) $(x+1)(x+2)(x+3)(x+4)=8$.

Here we observe that $2+3=1+4$, so that, if we multiply out $x+1$ with $x+4$ and $x+2$ with $x+3$, the two resulting quadratic expressions will have the same terms in x^2 as well as in x.

Thus we obtain $(x^2+5x+4)(x^2+5x+6)=8$
Putting $x^2+5x+4=y$, we have
$y(y+2)=8$ *i.e.*, $y^2+2y-8=0$ $\therefore$ $y=-4$ or 2

Now, if $y=x^2+5x+4=-4$, we have $x^2+5x+8=0$,

$$\text{whence } x=\frac{-5\pm\sqrt{-7}}{2}$$

Similarly, if $y=x^2+5x+4=2$, we have

$$x=\frac{-5\pm\sqrt{17}}{2}.$$

(*iv*) $(2x+1)(x^2-4)(2x-7)+27=0$

Here $(2x+1)(x-2)=2x^2-3x-2$
and $(2x-7)(x+2)=2x^2-3x-14$
$\therefore$ the given equation becomes

$$(2x^2-3x-2)(2x^2-3x-14)+27=0$$

Put $2x^2-3x-2=y$, so that $y(y-12)+27=0$
$\therefore$ $y^2-12y+27=0$ *i.e.*, $y=3$ or 9.

If $y=2x^2-3x-2=3$, we have $x=\frac{5}{2}$ and -1

and if $y=2x^2-3x-2=9$, we have $x=\dfrac{3\pm\sqrt{97}}{4}$

(*v*) $x^3-6x^2+11x-6=0$

Here 1 is easily seen to be a root, so that $x-1$ is a factor of the left-hand side.

Factorising, we have

$$(x-1)\ x^2-5x+6)=0 \quad i.e., \ (x-1)(x-2)(x-3)=0$$

$\therefore$ $x=1, 2, 3$ are the required roots.

1·7. Reciprocal Equations.

If an equation remains unchanged, when x is replaced by $\frac{1}{x}$, it is said to be a reciprocal equation. It is evident that in a reciprocal equation terms equidistant from the beginning and the end must have equal co-efficients either in magnitude or in magnitude as well as in sign.

Thus $2x^4+9x^3+14x^2+9x+2=0$.........(*i*)
$2x^3-3x^2-3x+2=0$........(*ii*)
$ax^4+bx^3+cx^2+bx+a=0$.......(*iii*)
$x^4+3x^3 \quad -3x-1=0$........(*iv*)
$ax^4+bx^3 \quad -bx-a=0$....... .(*v*)
$ax^3+bx^2-bx-a=0$.........(*vi*)

are all reciprocal equations. In (*i*), (*ii*) and (*iii*) terms equidistant from the beginning and the end have equal co-efficients in magnitude as well as in sign. In (*iv*) (*v*), they have co-efficients equal in magnitude but opposite in sign. It is also evident that in a reciprocal equation of even degree and of the type of (*iv*), (*v*) and (*vi*), the middle term must have its co-efficient zero, for zero alone can be its own negative.

Moreover, the roots of a reciprocal equation occur in reciprocal pairs. Hence, if the degree of a reciprocal equation be odd, one of its roots must be its own reciprocal, *i. e.*, one of the roots must be either $+1$ or -1.

The procedure for solving a reciprocal equation is illustrated by the following examples :—

(*i*) $2x^4+9x^3+14x^2+9x+2=0$

Grouping together terms which are equi-distant from beginning and end, we have

$$2(x^4+1)+9(x^3+x)+14=0$$

Dividing throughout by x^2, we get

$$2\left(x^2+\frac{1}{x^2}\right)+9\left(x+\frac{1}{x}\right)+14=0$$

Now put $x+\dfrac{1}{x}=y$, so that $x^2+\dfrac{1}{x^2}+2=y^2$, and obtain

$$2(y^2-2)+9y+14=0, \ i.\ e., \ 2y^2+9y+10=0$$

which gives $y=-2, -\frac{5}{2}$.

Taking $y=x+\dfrac{1}{x}=-2$, we have

$$x^2+2x+1=0, \text{ so that } x=-1, -1.$$

Again taking $y=x+\dfrac{1}{x}=-\frac{5}{2}$, we have

$$2x^2+5x+2=0, \text{ so that } x=-2, -\tfrac{1}{2}$$

$\therefore$ the required roots are $-1, -1, -2, -\frac{1}{2}$

(*ii*) $x^5-4x^4+x^3+x^2-4x+1=0$.

Here $x=-1$, satisfies the given equation, so that $x+1$ is a factor of the left-hand side and the equation can be written as

$$(x+1)(x^4-5x^3+6x^2-5x+1)=0$$

Removing the factor $x+1=0$, which gives $x=-1$, we have to solve the equation

$$x^4-5x^3+6x^2-5x+1=0$$

The procedure is now exactly the same as in (*i*) above and the roots of this equation are found to be $\dfrac{1\pm\sqrt{-3}}{2}$, $2\pm\sqrt{3}$.

Exercises

Solve the following equations :—

1. $\sqrt{\dfrac{x}{1-x}}+\sqrt{\dfrac{1-x}{x}}=\frac{13}{6}.$

2. $x^2-3x+2=2\sqrt{x^2-3x+2}.$

3. $\dfrac{x^2-1}{3}+\dfrac{1}{x^2}=1.$

4. $2(x^{\frac{1}{n}}+x^{-\frac{1}{n}})=5.$

5. $x^{\frac{1}{4}}+5x^{\frac{1}{2}}=22.$

6. $2^{x+1}+4^x=80.$

7. $(x-2)(x+3)(x+6)(x+1)+56=0.$

8. $(x+1)(2x+3)(2x+5)(x+3)=945.$ (*P. U.* 1942 *Supp.*)

9. $(x+a)(x+3a)(x+5a)(x+7a)=384a^4.$ (*C. U.*)

10. $6x^5-5x^4-24x^3+24x^2+5x-6=0.$

11. $x^4-2x^3-2x^2+2x+1=0.$

12. $6x^4+5x^3-38x^2+5x+6=0.$

13. $2x^4+x^3-6x^2+x+2=0.$

14. $2x^5+x^4-12x^3-12x^2+x+2=0.$

1·8. (*i*) Evaluate $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\ldots\ldots}}}}$

where dots indicate the unending continuation. (*P. U.* 1940)

Let $x=\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\ldots\ldots\ldots}}}}=\sqrt{6+x}$

$\therefore\ x^2=x+6$ *i. e.*, $x^2-x-6=0.$

This gives $x=3$ or -2.

Now -2 is found to be an extraneous root, so that the required value is 3.

(*ii*) Solve $(x-1)(x-2)(x-3)=5.4.3$

We can write this equation in the form

$$(x-1)(x-2)(x-3)=(6-1)(6-2)(6-3)$$

which shows that $x=6$ is a root.

The other roots may now be found out as usual. They are $\pm\sqrt{-11}$.

Revision Exercises

Solve the following equations :—

1. $(b-c)x^2+(c-a)x+(a-b)=0.$ (*P. U.* 1921)

2. $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{a-c}{x+a-c}+\dfrac{b-c}{x+b-c}.$

3. $(x-a+2b)^3-(x-2a+b)^3=(a+b)^3.$

4. $\dfrac{x+a}{x-b}+\dfrac{x+b}{x-a}=\dfrac{x^2-bx+3ab+b^2}{(x-a)(x-b)}.$

5. $\dfrac{x^3+3x}{3x^2+1}=3$ (*Hint.* - Apply Comp. Divi. and proceed).

6. $x^4+x^3-4x^2+x+1=0.$ (*P. U.* 1912)

7. $x^2+\dfrac{2}{3x}=\dfrac{13}{9}.$ (*P. U.* 1906)

8. $\sqrt{\dfrac{x}{x+16}}+\sqrt{\dfrac{x+16}{x}}=\dfrac{25}{12}$ (*P. U.* 1921)

9. $\dfrac{4x-1}{4x+1}+\dfrac{4x+1}{4x-1}=3\tfrac{1}{3}.$ (*P. U.* 1918)

10. $(x-7)(x-3)(x+1)(x+5)=1680.$

11. $\left(x-\dfrac{1}{x}\right)^2-3\left(x+\dfrac{1}{x}\right)=0.$ (*P. U.* 1914)

12. $\dfrac{x+\sqrt{x^2-2}}{x-\sqrt{x^2-2}}=\dfrac{x^3+1}{2}-x.$ (*P. U.* 1914)

13. $\dfrac{x^2}{6x-9}+\dfrac{6x-9}{x^2}=2.$

14. $x^{\frac{1}{3}}-13x^{\frac{1}{4}}=14.$

15. $(2x-7)(x^2-9)(2x+5)=91.$

16. $x^2-x+3\sqrt{2x^2-3x+2}=\dfrac{x}{2}+7.$

17. $\sqrt{2x^2-5x-3}+3\sqrt{2x+1}=\sqrt{2x^2+25x+12}.$

18. $\sqrt{5-x}+\sqrt{x+8}=5\sqrt{2-x}.$

19. $\dfrac{x-a}{b}+\dfrac{x-b}{a}=\dfrac{b}{x-a}+\dfrac{a}{x-b}.$

20. $\sqrt{4+4x-3x^2}-\sqrt{4+2x-6x^2}=\sqrt{8+6x-9x^2}$.

21. $(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}=a^{\frac{1}{3}}$. 22. $abx^2=(a+b)^2(x-1)$.

23. The area of a rectangle is 15 sq. ft. and the length is greater than the breadth by 2 ft. Find the length and breadth.

24. The sum of two numbers is 21 and their product is 110. Find them.

25. If 1 be a root of $x^2+x+b=0$, find the value of b.

26. What is the value of x, for which the expression x^2+4x+1 equals 6 ?

27. Nine times the square root of two-thirds of the number of elephants in a herd plus six times the square root of three-fifths of the remainder entered a forest and 24 remained behind. What was the number of elephants in the herd ?

(*Allahabad*, 1940)

CHAPTER II

Theory of Quadratic Equations

2·1. *A quadratic equation cannot have more than two distinc roots.*

If possible, let α, β, γ be three distinct roots of the quadratic equation $ax^2+bx+c=0$, where a is not zero.

$\therefore$ we have
$$a\alpha^2+b\alpha+c=0 \qquad \ldots(i)$$
$$a\beta^2+b\beta+c=0 \qquad \ldots(ii)$$
$$a\gamma^2+b\gamma+c=0 \qquad \ldots(iii)$$

From (ii) and (iii), by subtraction, we have
$$a(\beta^2-\gamma^2)+b(\beta-\gamma)=0,$$
or,
$$(\beta-\gamma)[a(\beta+\gamma)+b]=0,$$
Now
$$\beta-\gamma\neq 0, \text{ for } \beta\neq\gamma,$$
$$\therefore \quad a(\beta+\gamma)+b=0 \qquad \ldots(iv)$$

Similarly from (i) and (iii) since $\gamma\neq\alpha$, we have
$$a(\gamma+\alpha)+b=0 \qquad \ldots(v)$$

From (*iv*) and (*v*) by subtraction

$$a(\alpha-\beta)=0 \qquad \ldots(vi)$$

Now (*vi*) is impossible, since neither a nor $\alpha-\beta$ is zero. Hence our supposition is wrong and a quadratic equation cannot have more than two distinct roots.

Cor. If a quadratic equation is found to be satisfied by three different values of x, it will be satisfied by all the values of x.

For, if the quadratic equation $ax^2+bx+c=0$ is satisfied by α, β, γ, three different values of x, then from (*vi*) above, we must have $a=0$

Putting $a=0$ in (*iv*), we have $b=0$.
Putting $a=b=0$ in (*i*), we have $c=0$.

Thus the equation takes a unique form

$$0.x^2+0.x+0=0,$$

which is evidently satisfied by all values of x.

Def. An equation which is satisfied for all values of the variable is called an *identity*.

For example, $2x^2+5x+6=x^2$ is an equation for it is satisfied only by $x=-2$ and $x=-3$, whereas

$$(x+2)(x+3)=x^2+5x+6,$$

is an identity, for it is satisfied by all values of x. In order to distinguish between an equation and an identity, we frequently use the symbol $\equiv$ instead of $=$.

Thus $$(x+2)(x+3)\equiv x^2+5x+6.$$

It is important to note that *the co-efficients of like powers of x on both sides of an identity are equal.*

For let $a_0x^n+a_1x^{n-1}+\ldots\ldots+a_{n-1}x+a_n$

$$\equiv a_0'x^n+b_1'x^{n-1}+\ldots\ldots+a'_{n-1}x+a'_n.$$

Since this is satisfied by all values of x, we have on putting $x=0$, $a_n=a_n'$.

Thus the identity reduces to

$$a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x\equiv a_0'x^n+a_1'x^{n-1}+\ldots+a_{n-1}x..$$

$$i.e., \; a_0x^{n-1}+a_1x^{n-2}+\ldots+a_{n-1}\equiv a_0'x^{n-1}+a_1'x^{n-2}+\ldots+a'_{n-1}$$

Again putting $x=0$, we have $a_{n-1}=a'_{n-1}$.

Proceeding in a similar way, we can prove that $a_0=a_0'$, $a_1=a'_1$, $a_2=a'_2$ and so on, which proves the statement.

Ex. 1. Prove that the equation

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1,$$

is an identity.

Sol. The given equation is quadratic and is found to be satisfied by three different values of x, *viz*, a, b, c. Hence it must be an identity.

Ex. 2. Find the values of a, b, c, if

$$ax(x+1)+b(x+1)(x+2)+cx(x+2)=6x^2+13x+4,$$

is an identity.

Sol. If this equation is an identity, the co-efficient of like powers of x on both sides must be equal. Hence, equating the co-efficients of x^2, x and the absolute term on both sides, we have

$$a+b+c=6 \qquad ...(i)$$

$$a+3b+2c=13 \qquad ...(ii)$$

$$2b=4 \qquad ...(iii)$$

Solving these equations simultaneously, we get

$$a=1,\ b=2,\ c=3.$$

Exercises

1. Prove that the following are identities :—

(*i*) $a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^2.$

(*ii*) $(x-1)(x-2)+(x+2)(x-3)-(x-3)(x-1)=x^2-7.$

(*iii*) $f(a)\frac{(x-b)(x-c)}{(a-b)(a-c)}+f(b)\frac{(x-c)(x-a)}{(b-c)(b-a)}$

$$+f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)}=f(x),$$

where $f(x)$ denotes px^2+qx+r. (*P. U.* 1885)

2. Find the values of A, B, C, if

(*i*) $2x^2-3\equiv A(x-3)(x-4)+B(x-4)(x-2)+C(x-2)(x-3)$

(*ii*) $2x^2-3x+4 \equiv A(2-x)(2+3x)+B(2+3x)(1-x)$
$\qquad\qquad +C(1-x)(2-x).$

(*iii*) $x^3 \equiv A(x-a)(x^2+a^2)+B(x^2+a^2)+C(x-a)^2.$

3. If $(p-a)^3+(p-b)^3+(p-c)^3=3(p-d)^3$

$(q-a)^3+(q-b)^3+(q-c)^3=3(q-d)^3$

$(r-a)^3+(r-b)^3+(r-c)^3=3(r-d)^3$

then $(s-a)^3+(s-b)^3+(s-c)^3=3(s-d)^3.$

2·2. *To find the relations between the roots and co-efficients of a quadratic equation.*

If the roots of a quadratic equation $ax^2+bx+c=0$, be denoted by α, β, then we have

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad \beta=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

$\therefore$ sum of the roots $=\alpha+\beta$

$$=\frac{-b-\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b}{a}=-\frac{\text{co-efficient of } x}{\text{co-efficient of } x^2}.$$

Likewise, product of the roots $=\alpha\beta$

$$=\frac{-b+\sqrt{b^2-4ac}}{2a}\times\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}=\frac{b^2-(b^2-4ac)}{4a^2}$$

$$=\frac{c}{a}=\frac{\text{absolute term}}{\text{co-efficient of } x^2}.$$

Def. An expression is said to be symmetrical in α and β, if it remains unchanged, when α and β are interchanged. Thus $\alpha+\beta$, $\alpha\beta$, $\alpha^2+\beta^2$, $\alpha^2\beta+\alpha\beta^2$, are all symmetrical expressions in α and β.

If α, β denote the roots of a quadratic equation, the symmetric expressions $\alpha+\beta$ and $\alpha\beta$ can be evaluated at once. The following examples illustrate that any symmetric expressions in α, β can be evaluated with the help of $\alpha+\beta$ and $\alpha\beta$.

Ex. 3. If α, β denote the roots of $ax^2+bx+c,=0$ find the value of

$$(i)\ \alpha^2+\beta^2, \qquad (ii)\ \alpha^3+\beta^3 \qquad (iii)\ \frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$$

$$(iv)\ \alpha^4+\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3+\beta^4.$$

Sol. Here we have $\alpha+\beta=\dfrac{-b}{a}$, $\alpha\beta=\dfrac{c}{a}$.

Hence

$$(ii)\ \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(\frac{-b}{a}\right)^2-2.\frac{c}{a}$$

$$=\frac{b^2-2ac}{a^2}$$

$$(ii)\ \alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$$

$$=\left(\frac{-b}{a}\right)^3-3.\frac{c}{a}\left(\frac{-b}{a}\right)=\frac{3abc-b^3}{a^3}$$

$$(iii)\ \frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{3abc-b^3}{a^3}\cdot\frac{1}{c/a}$$

[Using the result (ii) above]

$$=\frac{3abc-b^3}{a^2c}.$$

$$(iv)\ \alpha^4+\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3+\beta^4$$

$$=(\alpha^4+\beta^4)+\alpha\beta(\alpha^2+\beta^2)+\alpha^2\beta^2$$

$$=[(\alpha^2+\beta^2)^2-2\alpha^2\beta^2]+\alpha\beta(\alpha^2+\beta^2)+\alpha^2\beta^2$$

$$=\left\{\left(\frac{b^2-2ac}{a^2}\right)^2-2.\frac{c^2}{a^2}\right\}+\frac{c}{a}\cdot\frac{b^2-2ac}{a^2}+\frac{c^2}{a^2}$$

$$=\frac{b^4-3ab^2c+a^2c^2}{a^4}.$$ [Using the result of (i) above]

2·21. *To form an equation whose roots are given.*

If α, β be the roots of the equation $ax^2+bx+c=0$ we may write the equation in the form

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

i. e., $$x^2-\left(\frac{-b}{a}\right)x+\frac{c}{a}=0$$

i. e., $$x^2-(\alpha+\beta)x+\alpha\beta=0$$

Thus the equation whose roots are known may be written as

x^2-(sum of the roots)$+$product of the roots$=0$.

Otherwise thus :

Since α is a root of the required equation, $x-\alpha$ must be a factor of the left-hand side. Similarly $x-\beta$ must be a factor. Also since the equation is quadratic, the left-hand side cannot have any other factor containing x. Hence the required equation must be

$$(x-\alpha)(x-\beta)=0$$

i. e. $$x^2-(\alpha+\beta)x+\alpha\beta=0.$$

Ex. 4. If α, β be the roots of $ax^2+bx+c=0$, form an equation whose roots are α^2, β^2.

Sol. Here $\alpha+\beta=\dfrac{-b}{a}$, $\alpha\beta=\dfrac{c}{a}$

Hence, sum of the roots of the required equation

$$=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{b^2-2ac}{a^2}$$

Also their product $=\alpha^2\beta^2=\dfrac{c^2}{a^2}$

Therefore the required equation is

$$x^2-\frac{b^2-2ac}{a^2}x+\frac{c^2}{a^2}=0,$$

or $$a^2x^2-(b^2-2ac)x+c^2=0.$$

Otherwise thus :—

The given equation is

$$ax^2+bx+c=0$$

Let y be a root of the required equation,

then $y=x^2$

or $x=\sqrt{y}$

The equation in x becomes

$$a(\sqrt{y})^2+b\sqrt{y}+c=0, \text{ or } ay+c=-b\sqrt{y}$$

Removing radicals, we get

$$(ay+c)^2=b^2y$$

or $\quad a^2y^2+(2ac-b^2)y+c^2=0.$

The required equation may be written in y or in x. If y is changed into x, this becomes

$$a^2x^2+(2ac-b^2)x+c^2=0,$$

whose roots are the squares of the roots of the original equation.

Ex. 5. Form an equation, each of whose roots is greater than the corresponding root of the equation $x^2+2x-3=0$, by 5.

Sol. If α, β are the roots of the original equation, $\alpha+5$ and $\beta+5$ will be the roots of the required equation.

Also $\alpha+\beta=-2$, $\alpha\beta=-3$

$\therefore$ Sum of the roots of the required equation

$$=(\alpha+5)+(\beta+5)=(\alpha+\beta)+10=-2+10=8$$

Similarly their product $=(\alpha+5)(\beta+5)$

$$=\alpha\beta+5(\alpha+\beta)+25=-3+5(-2)+25=12$$

Hence the required equation is $x^2-8x+12=0$

Otherwise thus : –

Let y denote a root of the required equation,
so that $\quad y=x+5$ or $x=y-5$

Hence the equation in x becomes

$$(y-5)^2+2(y-5)-3=0$$

i. e., $\quad y^2-8y+12=0$

The required equation may be written in y or in x. If y is changed into x, the required equation is

$$x^2-8x+12=0.$$

Ex. 6. Find the condition that one root of

$$ax^2+bx+c=0,$$

may be the square of the other. (*P. U.* 1916)

Sol. Here if α be one of the roots, the other will be α^2.

$$\therefore \quad \alpha+\alpha^2=\text{sum of the roots}=\frac{-b}{a} \qquad \ldots(i)$$

and $\alpha.\alpha^2=\alpha^3=$product of the roots$=\dfrac{c}{a}$ $\qquad \ldots(ii)$

$$\text{From } (ii),\ \alpha=\left(\frac{c}{a}\right)^{\frac{1}{3}}$$

$$\therefore \text{ from } (i),\ \left(\frac{c}{a}\right)^{\frac{1}{3}}+\left(\frac{c}{a}\right)^{\frac{2}{3}}=\frac{-b}{a}$$

$$i.e. \quad a^{\frac{2}{3}}c^{\frac{1}{3}}+a^{\frac{1}{3}}c^{\frac{2}{3}}+b=0$$

$$\text{or} \quad a^{\frac{1}{3}}c^{\frac{1}{3}}\left(a^{\frac{1}{3}}+c^{\frac{1}{3}}\right)+b=0,$$

which is the required condition.

Exercises

1. Form the quadratic equation whose roots are

(*i*) 2, 3, (*ii*) -3, 4, (*iii*) 5, $\frac{1}{5}$, (*iv*) $\sqrt{2}$, $\sqrt{3}$,

(*iv*) $2+\sqrt{3}$, $2-\sqrt{3}$, (*vi*) $\dfrac{1}{3+\sqrt{5}}$, $\dfrac{1}{3-\sqrt{5}}$.

(*vii*) $\dfrac{\sqrt{m}}{\sqrt{m}\pm\sqrt{m-n}}$. (*P. U.* 1925)

2. Find the condition that the roots of
$$x^2-px+y=0$$
may be (*i*) equal in magnitude but opposite in sign (*ii*) reciprocals of each other.

3. If α, β be the roots of $ax^2+bx+c=0$, evaluate

(*i*) $\sqrt{\dfrac{\alpha}{\beta}}+\sqrt{\dfrac{\beta}{\alpha}}$, (*ii*) $(\alpha-\beta)^2$, (*iii*) $\alpha^2-\alpha\beta+\beta^2$,

(*iv*) $\dfrac{\alpha^4+\beta^4}{(\alpha-\beta)^2}$, (*v*) $(p+\alpha^2)(p+\beta^2)$, (*vi*) $(p-\alpha^2)(p+\beta^2)$,

(*vii*) $\left(\dfrac{\alpha}{\beta}-\dfrac{\beta}{\alpha}\right)^2$ (*Allahabad* 1940)

(*viii*) $\dfrac{\alpha^2}{\beta^4}+\dfrac{\beta^2}{\alpha^2}$, (*ix*) $\dfrac{\alpha^5-\beta^5}{\alpha^4-\beta^4}$, (*x*) $\dfrac{1}{(a\alpha+b)^4}+\dfrac{1}{(a\beta+b)^2}$.

4. If α, α^2 are tbe roots of $ax^2+bx+c=0$, find the value of c, when $a=8$, $b=-30$. (*P. U.* 1931)

5. If α, β be the roots of $ax^2+bx+c=0$, form the quadratic equation whose roots are

(*i*) $\alpha+\beta$, $\frac{1}{\alpha}+\frac{1}{\beta}$, (*P. U.* 1898)

(*ii*) $\alpha^2+\beta^2$, $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$, (*iii*) $\frac{1-\alpha}{1+\alpha}$, $\frac{1-\beta}{1+\beta}$,

(*iv*) $(\alpha+\beta)^2$, $(\alpha-\beta)^2$, (*v*) $\alpha^3+\beta^3$, $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$ (*P.U.* 1912)

(*vi*) $\frac{\alpha}{1+\beta}$, $\frac{\beta}{1+\alpha}$. (*P. U.* 1926)

(*vii*) $\alpha+\frac{1}{\alpha}$, $\beta+\frac{1}{\beta}$ (*P. U.* 1927)

6. If the roots of $ax^2+bx+c=0$, be in the ratio $k:1$, prove that $ac(1+k^2)=k\ b^2-2ac)$,

7. If α, β be the roots of $x^2-px+q=0$, and γ, δ those of $x^2-p'x+q'=0$, find the value of

$$(\alpha-\gamma)(\beta-\delta)+(\alpha-\delta)(\beta-\gamma)$$

8. Find the value of $\frac{(p+\alpha)(p+\beta}{(q-\alpha^2)(q-\beta^2)}$, where α, β are the roots of $x^2+px+q=0$.

9. If α, β are the roots of $x^3-\gamma x+\delta=0$, form the equation whose roots are γ, δ.

10. If each of the roots of $x^2-px+q=0$, differs from the corresponding root of $x^2-qx+p=0$, by the same quantity then $p+q+4=0$.

2·3 *To discuss the nature of the roots of a quadratic equation.*

We have seen that the roots of the equation $ax^2+bx+c=0$, are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}$$

(*i*) If b^2-4ac is a positive quantity the equation will have two real and distinct roots. Further, if b^2-4ac is a perfect square, the two roots will be rational, otherwise they will be irrational.

Note that if a, b, c are rational, either both or none of the roots will be rational. In other words, in an equation with rational co-efficients, surd roots occur in conjugate pairs *i. e.*, if $\alpha+\sqrt{\beta}$ is a root, then so is $\alpha-\sqrt{\beta}$.

(*ii*) If b^2-4ac is equal to zero, both the roots become equal to $-\dfrac{b}{2a}$. In this case the equation is satisfied only by a single value of x; and we say that in this case the equation has two equal roots.

(*iii*) If b^2-4ac is a negative quantity $\sqrt{b^2-4ac}$ is not a real number; for there is no real number whose square is equal to the negative quantity b^2-4ac. Thus the equation has two imaginary roots.

Note that if a, b, c are real, either both or none of the roots will be imaginary.

In other words, in an equation with real co-efficients, imaginary roots occur in conjugate pairs, *i. e.*, if $a+\sqrt{-\beta}$ is a root, so is $a-\sqrt{-\beta}$ where β is positive.

Note :—(*i*) If $b^2-4ac<0$, then the expression ax^2+bx+c has the same sign as that of a or c, for all real values of x.

For $$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$=a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right].$$

Now $\left(x+\dfrac{b}{2a}\right)^2$ is positive for all real values of x. Also since $b^2-4ac<0$, we have $4ac-b^2>0$.

$\therefore \left(x+\dfrac{b}{2a}\right)^2+\dfrac{4ac-b^2}{4a^2}$ is positive for all real values of x.

Hence ax^2+bx+c has, for all real value of x, the same sign as that of a.

Moreover, if $b^2-4ac<0$, ac must be positive, *i. e.*, a and c have the same sign.

(*ii*) If $b^2-4ac=0$, then the expression ax^2+bx+c, after division by a is a perfect square.

For
$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$
$$=a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$
$$=a\left(x+\frac{b}{2a}\right)^2, \text{ since } b^2-4ac=0.$$

(*iii*) The quantity b^2-4ac, which is utilised in finding the nature of the roots of a quadratic equation is called the **Discriminant** of the equation.

Thus we say that the roots of a quadratic equation are

(*i*) real, distinct and rational, if the discriminant is positive and a perfect square ;

(*ii*) real, distinct and irrational, if the discriminant is positive but not a perfect square ;

(*iii*) real and equal, if the discriminant is zero ;

and (*iv*) imaginary, if the discriminant is negative.

2·31. Signs of roots

To find the conditions that the equation $ax^2+bx+c=0$, whose roots are supposed to be real, may have its roots

(*i*) both positive (*ii*) both negative

and (*iii*) one positive and one negative.

(*i*) If both the roots are positive, their sum as well as their product must be positive. Hence $\frac{-b}{a}$ and $\frac{c}{a}$ are both positive *i. e.*, a and c have the same sign, while b has the sign opposite to that of a or c.

(*ii*) If both the roots are negative, their sum is negative, while their product is positive. Therefore in this case $\frac{-b}{a}$ is negative while $\frac{c}{a}$ is positive, so that a, b, c have the same sign.

(*iii*) If one of the roots is positive and the other negative, their product c/a is negative. Hence c and a must have opposite signs.

2·32. *To find the condition that the equation*

$$ax^2+bx+c=0$$

may have (*i*) *one root zero,* (*ii*) *both roots zero.*

(*i*) If one of the roots is zero, the product of the two roots c/a, must be zero.

Hence in this case $c=0$.

(*ii*) If both the roots are zero, then their sum $-b/a$ as well as their product c/a must be zero.

Hence in this case $b=c=0$.

2·33. Def. A quantity is said to tend to infinity if it goes on increasing and there is no limit to its increase, so that it becomes larger than any quantity however large. If x tends to infinity, we express this fact by writing $x\to\infty$ or by saying that x is infinite.

Note that (*i*) ∞ is not a number,

and (*ii*) $x=\infty$ has no meaning.

Similarly if the numerical value of a quantity x decreases without limit, so that it can be made as small as we please, we say that x tends to zero and write $x\to 0$. We also say that x is a *vanishing* quantity.

The student must convince himself that

(*i*) If $x\to\infty$ and a is any positive number, then $x\pm a$, $x\times a$, x/a also tend to infinity, while $a/x\to 0$.

(*ii*) If $x/y \to 0$, then $x \to 0$, if y is a fixed number and $y \to \infty$, if x is a fixed number.

(*iii*) If $x/y \to \infty$: then $x \to \infty$, if y is a fixed number and $y \to 0$, if x is a fixed number.

2·331. *To find the condition that the equation*

$$ax^2+bx+c=0, \textit{ may have}$$

(1) *one root infinite,* (2) *both roots infinite.*

Let us form the equation whose roots are the reciprocals of the roots of the equation

$$ax^2+bx+c=0 \qquad \dots \quad (i)$$

This equation is found to be

$$cx^2+bx+a=0 \qquad \dots \quad (ii)$$

Now one root of (*i*) will $\to \infty$, if the corresponding root of (*ii*) $\to 0$.

But one root of (*ii*) will tend to 0 if $a \to 0$. This is therefore the condition that one root of (*i*) be infinite.

Again both the roots of (*i*) will tend to ∞, if both roots of (*ii*) tend to 0, which will be the case, if $a \to 0$ and $b \to 0$. These are therefore the conditions that both roots of (*i*) should be infinite.

Ex. 7. Find the nature of the roots of the equations

(1) $2x^2-5x+4=0$.

(2) $4x^2-12x+9=0$.

(3) $6x^2-7x+1=0$.

(4) $6x^2-7x-1=0$.

Sol. (1)

Discriminant$=b^2-4ac=(-5)^2-4\times 2\times 4$
$=25-32=-7$, which is negative.

Hence the roots are imaginary.

(2) Here $b^2-4ac=(-12)^2-4\times 4\times 9=0$.

$\therefore$ the roots are real and equal.

(3) Here $b^2-4ac=(-7)^2-4\times 6\times 1=25$, which is positive and a perfect square.

$\therefore$ the roots are real, distinct and rational.

(4) Here $b^2-4ac=(-7)^2-4\times6\times(-1)=73$, which is positive but not a perfect square.

$\therefore$ the roots in this case are real, distinct and irrational.

Ex. 8. For what value of m will the equation $(m+1)x^2+2(m+3)x+2m+3=0$, have equal roots. (*P. U.* 1943)

Sol. Here the discriminant b^2-4ac

$$\begin{aligned}&=4(m+3)^2-4(m+1)(2m+3)\\&=4[(m^2+6m+9)-(2m^2+5m+3)]\\&=4(6+m-m^2).\end{aligned}$$

If the roots are equal, $4(6+m-m^2)=0$,

$\therefore$ $(m-3)(m+2)=0$, *i.e.*, $m=-2, 3$.

Exercises

Examine the nature of the roots of the following equations :—

1. $x^2-5x-2=0$ (*P. U.* 1943).
2. $2x^2-4x+5=0$.
3. $(x-a)(x-b)=c^2$.
4. $3x^2+7x+2=0$.
5. Find k such that the equation

$$(4-k)x^2+(2k+4)x+(8k+1)=0,$$

may have equal roots. (*P. U.* 1920)

6. Show that the roots of

$$(x-b)(x-c)+(x-c)(x-a)+(x-a)(x-b)=0,$$

are real and that they cannot be equal, unless $a=b=c$. (*P. U.* 1929)

7. Show that if the roots of

$$x^2(a^2+a'^2)+2x(ab+a'b')+b^2+b'^2=0,$$

are real, they are equal also.

8. Discuss the nature of the roots of

$$(\alpha^2+\beta^2)x^2+2(\alpha^3+\beta^3)x+\alpha^4+\beta^4=0,$$

where α, β are real. (*Allahabad*, 1930)

9. Show that the roots of

(*i*) $x^2-2x\left(m+\dfrac{1}{m}\right)+3=0$. (*P. U.*)

and of (*ii*) $(b^2-4ac)x^2+4(a+c)x-4=0$.
are real. (*P. U.*)

10. Prove that the roots of

$$(p+q)x^2-2px+(p-q)=0,$$

are rational. (*P. U.*)

11. Prove that, if the roots of $ax^2+2bx+c=0$, are real and unequal, then the roots of the equation

$$x^2+2(a+c)x+(a^2+2b^2+c^2)=0$$

are imaginary.

12. Show that the roots of the equation

$$(a-b)x^2+2(a+b)x-(a-b)=0$$

are real.

13. Find the condition that

$$m^2x^2+2x(mc-2a)+c^2=0$$

may have equal roots.

14. Find the value of k, for which the expression

$$x^2-2x(1+3k)+7(3+2k)$$

is a perfect square.

15. Show that the expression x^2-5x+7 is positive for all real values of x.

2·4. Imaginary Numbers.

A quantity $\sqrt{x}$ is called real or imaginary according as x is positive or negative. Thus $\sqrt{7}$, $\sqrt{4}$ are real and $\sqrt{-7}$ and $\sqrt{-4}$ are imaginary quantities. The imaginary number $\sqrt{-1}$, for sake of brevity, is denoted by i (read iota).

All imaginary numbers can be expressed as the product of a real number and the imaginary number i. For if $\sqrt{-x}$, be an imaginary number, where x is positive, we may write $\sqrt{-x}=\sqrt{x}.\sqrt{-1}=\sqrt{x}\,.\,i$, where $\sqrt{x}$ is real.

A quantity of the form $a+\sqrt{-b^2}$, where a, b are real, is called a complex number. It may also be written as $a+bi$; a is called the real part and bi, the imaginary part. If the real part is zero, the number is purely imaginary and if the imaginary part is zero, the number is purely real.

No real number can equal a complex number.

For, if possible, let $a+ib=x$, where a, b, x are all real.

We may then write $x-a=ib$

Squaring both sides $(x-a)^2=-b^2$

$$(\because i^2=\sqrt{-1}\times\sqrt{-1}=-1)$$

i.e., $(x-a)^2+b^2=0$, which is possible only, if $x=a$ and $b=0$.

Further, *a complex number will be zero, if and only if its real and imaginary parts are separately zero.*

For, if $a+ib=0$, we have $a=-ib$
squaring $a^2=i^2b^2=-b^2$
$\therefore$ $a^2+b^2=0$,
which is possible only if $a=b=0$.

Two complex numbers will be equal, if and only if their real and imaginary parts are separately equal.

For, if $a+ib=c+id$

we have $(a-c)+i(b-d)=0$

$\therefore$ $a-c=0$ and $b-d=0$,

giving $a=c$, $b=d$.

2·41. Two complex numbers are said to be **conjugate**, if they differ only in the sign of their imaginary parts. If one of these numbers is denoted by $a+ib$, the other will be $a-ib$.

The sum as well as the product of a pair of conjugate complex numbers is real.

For, if $a\pm ib$ are the two conjugate complex numbers, we have,

$$(a+ib)+(a-ib)=2a$$

and $(a+ib)(a-ib)=a^2-i^2b^2=a^2-(-b^2)=a^2+b^2$.

2·42. Powers of *i*.

Any integral power of i has one of the four values ±1, $\pm i$.

We have $i=i$; $i^2=-1$; $i^3=i^2\times i=-i$ and $i^4=(i^2)^2=(-1)^2=+1$.

Generally, any integer can be put in one of the forms $4n$, $4n+1$, $4n+2$, $4n+3$, according as it leaves a remainder 0, 1, 2 3 on being divided by 4. Now

$$i^{4n}=(i^4)^n=1$$
$$i^{4n+1}=(i^4)^n.i=i$$
$$i^{4n+2}=(i^4)^n.i^2=-1$$
$$\text{and } i^{4n+3}=(i^4)^n.i^3=-i$$

Hence the statement.

2·43. Cube roots of Unity. If x be a cube root of unity, we have

$$x^3=1, \quad i\,e., x^3-1=0$$
$$\therefore \quad (x-1)(x^2+x+1)=0$$

Hence, either $x-1=0$, giving $x=1$

$$\text{or } x^2+x+1=0, \text{ giving } x=\frac{-1\pm\sqrt{-3}}{2}$$

$\therefore$ 1, $\dfrac{-1+\sqrt{-3}}{2}$, $\dfrac{-1-\sqrt{-3}}{2}$, are the three cube roots of unity, the last two being imaginary.

An interesting fact about the two imaginary cube roots of unity is that *each of them is the square of the other.*

$$\text{Thus} \quad \left(\frac{-1+\sqrt{-3}}{2}\right)^2=\frac{1+(-3)-2\sqrt{-3}}{4}=\frac{-1-\sqrt{-3}}{2}$$

$$\text{and} \quad \left(\frac{-1-\sqrt{-3}}{2}\right)^2=\frac{1+(-3)+2\sqrt{-3}}{4}=\frac{-1+\sqrt{-3}}{2}$$

Hence, if we denote one of the imaginary cube roots of unity be w, the other may be denoted by w^2.

Thus 1, w, w^2 are the three cube roots of unity.

It may be pointed out that

$$w+w^2=-1 \text{ and } w.w^2=1.$$

This follows either by actual substitution or from the fact that w, w^2 are the roots of the equation $x^2+x+1=0$.

2·44. Ex. 9. Factorise a^2+b^2.

Sol. We have $a^2+b^2=a^2-(-b^2)=a^2-i^2b^2$
$=(a+ib)(a-ib).$

Ex. 10. Find the square root of $3+4i$

Sol. Let $\sqrt{3+4i}=x+iy$

$\therefore \quad 3+4i=(x+iy)^2=x^2+i^2y^2+2i.xy$

i. e., $\quad 3+4i=x^2-y^2+i.2xy \quad (\because i^2=-1)$

Hence, equating real and imaginary parts,
$x^2-y^2=3$ and $2xy=4.$

Eliminating y, $x^2-\dfrac{4}{x^2}=3$, *i. e.,* $x^4-3x^2-4=0$,

which gives $x^2=4$ or -1. We reject -1, as x is real. Hence $x=\pm 2$, correspondingly $y=\pm 1$.

$\therefore \quad \sqrt{3+4i}=2+i$ or $-2-i$.

Ex. 11. Evaluate x^3+3x^2+3x+1, if $x=-1+2i$.

Sol. From $x=-1+2i$, we have $x+1=2i$

Cubing both sides, $(x+1)^3=8i^3$

$\therefore \quad x^3+3x^2+3x+1=-8i.$

Ex. 12. Prove that $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^5+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^5=-1$

(*P. U.* 1898)

Sol. The expression on the L.H.S.

$=w^5+w^{10}=w^2.w^3+w.w^9=w^2.1+w.1$

$=w+w^2=-1.$

Exercises.

1. Multiply (*i*) $5+3i$ by $3+4i$.
   (*ii*) $\sqrt{-7}+\sqrt{-3}$ by $\sqrt{-5}+\sqrt{-2}$.

2. Simplify (*i*) $\dfrac{3+4i}{5+2i}$, (*ii*) $\dfrac{2+3i}{4+7i}+\dfrac{2-3i}{4-7i}$

3. Find the square root of
   (*i*) $2i$, (*ii*) $-2i$, (*iii*) $-7+24i$, (*iv*) $-7-24i$.
   (*Delhi*, 1943)

4. Find the cube root of -1.

5. Prove that (*i*) $\left(\frac{-1+\sqrt{-3}}{2}\right)^{16}+\left(\frac{-1-\sqrt{-3}}{2}\right)^{16}=-1$.

(*ii*) $\left(\frac{-1+\sqrt{-3}}{2}\right)^{15}+\left(\frac{-1-\sqrt{-3}}{2}\right)^{15}=2$.

(*iii*) $\left(\frac{-1+\sqrt{-3}}{2}\right)^{n}+\left(\frac{-1-\sqrt{-3}}{2}\right)^{n}=2$

or -1, according as the integer n is or is not a multiple of 3.

6. Find the value of

(*i*) $x^4+4x^3+6x^2+4x+2$, when $x=-1+\sqrt{-1}$,

(*ii*) x^3-6x^2+12x, when $x=2+3\sqrt{-1}$,

7. Show that if $a+ib=3+2i$, then $a-ib=3-2i$.

8. Show that (*i*) $(1+2w+w^2)(1+w+2w^2)=1$.

(*ii*) $(1-w)(1-w^2)=3$.

(*iii*) $(1-w-w^2)^4=16$

9. Find all the cube roots of

(*i*) 8, (*ii*) 27, (*iii*) any real number a^3.

10. Show that $x^3-y^3=(x-y)(x-wx)(x-w y)$.

2·5. Limits of Quadratic Expressions.

Ex. 13. If x is real, prove that the expression $x^2-12x+40$ can not be less than 4.

Sol. Let $x^2-12x+40=y$

$\therefore$ $x^2-12x+(40-y)=0$.

Now x is real, so that the discriminant is positive.

$\therefore$ $(-12)^2-4(40-y)\geqslant 0$.

i. e., $144-160+4y\geqslant 0$, *i. e.*, $4y\geqslant 16$ or $y\geqslant 4$.

$\therefore$ $y=x^2-12x+40$ cannot be <4, if x is real.

Ex. 14. If x is real, prove that the expression $\frac{x^2-4x+4}{x-1}$ cannot lie between 0 and -4.

Sol. Let $\frac{x^2-4x+4}{x-1}=y$, so that $x^2-(y+4)x+(y+4)=0$.

Now x is real so that the discriminant is positive.

$\therefore \quad (y+4)^2-4(y+4) \geqslant 0.$
i. e., $(y+4)(y+4-4) \geqslant 0.$
i. e., $y(y+4) \geqslant 0.$

Hence either both of y and $y+4$ are positive, which is the case when y is greater than 0, or both of them are negative, which is the case when y is less than -4.

$\therefore$ the given expression, for real values of x is either greater than 0 or less than -4.

Ex. 15. If x is real, find the limits between which the expression $\dfrac{11x^2+12x+6}{x^2+4x+2}$ lies.

Sol. Let $\dfrac{11x^2+12x+6}{x^2+4x+2}=y$, so that

$$(11-y)x^2+4(3-y)x+6-2y=0.$$

Now x is real.

$\therefore$ the discriminant $16(3-y)^2-4(11-y)(6-2y)$ is positive
i. e., $2(3-y)^2-(11-y)(3-y)$ is positive
i. e., $(3-y)(-5-y)$ is positive
i. e., $(y-3)(y+5)$ is positive.

$\therefore$ either both of $y-3$ and $y+5$ are positive, or both of them are negative.

Both of them will be positive, if y is greater than 3 and both of them will be negative, if y is less than -5.

$\therefore$ the given expression can have any real value except between -5 and 3

Exercises.

If x is real, prove that

1. $\dfrac{2x-7}{2x^2-2x-5}$ can not lie between 1 and $\frac{1}{11}$.

2. $\dfrac{x^2-x+1}{x^2+x+1}$ always lies between 3 and $\frac{1}{3}$.

3. $\dfrac{x^2+2x+1}{x^2+2x+7}$ always lies between 0 and 1.

4. $x+\dfrac{1}{x}$ cannot lie between 2 and -2.

5 $\dfrac{x^2-3x+4}{x^2+3x+4}$ always lies between 7 and $\frac{1}{7}$.

6. $\dfrac{x^2+2x-11}{2x-6}$ cannot lie between 2 and 6.

7. $\dfrac{x^2+x-2}{1+x-2x^2}$ can have any real value.

8. $\dfrac{4x^2+2x-1}{x^2+6x+3}$ can have any real value.

9. $4+5x-2x^2$ can never be greater than $7\frac{1}{8}$.

10. $\dfrac{x^2+34x-71}{x^2+2x-7}$ can have all numerical values except such as lie between 5 and 9. (*Delhi* 1937)

11. $\dfrac{(x-a)(x-c)}{x-b}$ is capable of assuming all values provided a, b, c are in ascending order of magnitude.

2·6. *To find the condition that the two quadratic equations*

$$ax^2+bx+c=0,\ a'x^2+b'x+c'=0,$$

have a common root.

Let α be a number which satisfies both the equations so that

$$a\alpha^2+b\alpha+c=0$$

and $$a'\alpha^2+b'\alpha+c'=0.$$

Therefore by the rule of cross-multiplication

$$\frac{\alpha^2}{bc'-b'c}=\frac{\alpha}{ca'-c'a}=\frac{1}{ab'-a'b}$$

Eliminating α, we get

$$(bc'-b'c)(ab'-a'b)=(ca'-c'a)^2,$$

which is the required condition.

Cor. When the two quadratic equations have a common root, it must be $\dfrac{bc'-b'c}{ca'-c'a}$ or $\dfrac{ca'-c'a}{ab'-a'b}$, these two expressions being equal to each other.

2·61. *To find the conditions that the two quadratic equations*

$$ax^2+bx+c=0 \text{ and } a'x^2+b'x+c'=0,$$

may have both roots common.

Let α, β be the roots of the first equation so that α, β are also the roots of the second. Hence

$$\alpha+\beta=\frac{-b}{a}=\frac{-b'}{a'} \text{ and } \alpha\beta=\frac{c}{a}=\frac{c'}{a'}$$

These give $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$

Hence, if two quadratic equations have the same roots, the corresponding co-efficients in them must be proportional.

Ex. 16. The equations

$$x^2+ax+bc=0 \text{ and } x^2+bx+ca=0,$$

have a common root, show that their other roots satisfy the equation $x^2+cx+ab=0$. (*P. U.* 1923)

Sol. Let α be the root common to both the equations, so that $\alpha^2+a\alpha+bc=0$ and $\alpha^2+b\alpha+ca=0$.

By subtraction $\alpha(a-b)+c(b-a)=0$,

$$i.\ e.,\quad (a-b)(\alpha-c)=0$$

$$\therefore\quad \alpha=c,\quad \because\ a-b\neq 0$$

Thus c is the common root and we have

$$c^2+ac+bc=0$$

$$i.\ e.,\quad a+b+c=0 \qquad \ldots(i)$$

Now product of the roots of the first equation $=bc$ and one of the roots $=c$,

$\therefore$ the other root is $\frac{bc}{c}=b$.

Similarly a is the other root of the second equation. Thus the required equation is

$$x^2-(a+b)x+ab=0$$

or $\quad x^2+cx+ab=0, \quad \because\ c=-(a+b)$, from (i)

Ex. 17. Determine the value of m for which

$$3x^2+4mx+2=0 \text{ and } 2x^2+3x-2=0,$$

have a common root. (*Calcutta* 1934)

Sol. Let α be a root common to both the equations, so that

$$3\alpha^2+4m\alpha+2=0 \text{ and } 2\alpha^2+3\alpha-2=0$$

Hence $\quad \frac{\alpha^2}{-8m-6}=\frac{\alpha}{4+6}=\frac{1}{9-8m}$

Eliminating α, we have

$$(8m+6)(8m-9)=100$$

i. e., $$32m^2-12m-77=0, \qquad \therefore m=\tfrac{7}{4}, -\tfrac{11}{8}.$$

Ex. 18. Find the condition that a root of $ax^2+bx+c=0$, may be equal to twice a root of $a'x^2+b'x+c'=0$.

Sol. Let α be a root of the first equation, so that 2α is a root of the second. Hence

$$a\alpha^2+b\alpha+c=0 \text{ and } 4a'\alpha^2+2b'\alpha+c'=0$$

$$\therefore \quad \frac{\alpha^2}{bc'-2b'c}=\frac{\alpha}{4ca'-c'a}=\frac{1}{2ab'-4a'b}$$

Eliminating α, the required condition is

$$(4ca'-c'a)^2=2(ab'-2a'b)(bc'-2b'c)$$

Exercises

1. If $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$, have a common root, prove that this common root will also satisfy the equation $(a+\lambda a')x^2+(b+\lambda b')x+(c+\lambda c')=0$ where λ is any constant.

2. If the equations $x^2+px+q=0$, and $x^2+rx+s=0$, have a common root, show that it is either $\dfrac{ps-qr}{q-s}$ or $\dfrac{q-s}{r-p}$. (*P. U.* 1934)

3. Prove that the condition that a root of $ax^2+bx+c=0$, may be the negative of a root of $a'x^2+b'x+c'=0$, is

$$(bc'+b'c)(ab'+a'b)+(ca'+c'a)^2=0.$$

4. Prove that the condition that a root of $ax^2+bx+c=0$, may be the negative reciprocal of a root of $a'x^2+b'x+c'=0$, is that $$(cc'-aa')^2+(ba'+b'c)(ab'+bc')=0.$$

5. Prove that the condition that a root of $ax^2+bx+c=0$ may be the reciprocal of a root of $a'x^2+b'x+c'=0$, is that

$$(b'c-a'b)(bc'-ab')=(aa'-c'c)^2.$$

6. If every pair of the equations

$$x^2-ax+bc=0,\ x^2-bx+ca=0 \text{ and } x^2-cx+ab=0$$

has a common root, prove that $a+b+c=0$.

7. The equation $ax^3+bx+c=0$ is unaltered when each of the co-efficients in the equation is increased by the same quantity, show that $x^2+x+1=0$. (*P. U.* 1917)

8. The equation $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$, have a common root, and the second of these has equal roots, prove that $2(ac'-a'b)=bb'$.

9. One root of $ax^2+bx+c=0$, is less than the other by 2 ; prove that $4a^2=b^2-4ac$.

10. Show that the equations

$$x^2+px+q=0 \text{ and } x^2+qx+p=0,$$

will have a common root, if $p=q$ or $p+q+1=0$:

11. If the equations $x^2+px+q=0$ and $x^2+qx+p=0$ have a common root, prove that their other roots satisfy the equation $x^2+x+pq=0$.

12. If the roots of $x^2+px+q=0$ exceed those of $x^2+p'x+q'=0$ by the same quantity, prove that

$$p^2-p'^2=4(q-q')^2$$

CHAPTER III

Simultaneous Equations

Equations containing more than one variables are known as *simultaneous* equations. The solution of such an equation is generally obtainable, when the number of equations given is equal to the number of variables involved. The different artifices that can be usefully employed are illustrated by the solved examples given below.

3 1. When one of the variable can be conveniently solved in terms of the other.

Examples :—(*i*) $x^2+y^2=130$,

$x-y=2$;

From the second equation, $y=x-2$.

Substituting in the first, we have

$$x^2+(x-2)^2=130$$

i. e., $2x^2-4x=126$ or $x^2-2x-63=0$

This gives $x=9$ or -7.

Whence from the second equation we get

$$y=7 \text{ or } -9.$$

$\therefore$ $x=9, y=7$, and $x=-7, y=-9$ are the two solutions.

(*ii*) $\dfrac{1}{x}-\dfrac{1}{y}=1,\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=13.$

From the first $\dfrac{1}{y}=\dfrac{1}{x}+1,$

$\therefore$ from the second $\dfrac{1}{x^2}+\left(\dfrac{1}{x}-1\right)^2=13$ or $\dfrac{2}{x^2}-\dfrac{2}{x}=12$

i. e. $6x^2+x-1=0,$

$\therefore$ $x=\frac{1}{3}$ or $-\frac{1}{2}$.

and correspondingly, $y=\frac{1}{2}$ or $-\frac{1}{3}$.

(*iii*) $\dfrac{x+y}{1-xy}=3,\ \dfrac{x-y}{1+xy}=\dfrac{1}{3}.$ (*Calcutta* 1903)

From the first equation $1-xy=\frac{1}{3}x+\frac{1}{3}y$

and put the second $1+xy=3x-3y$

$\therefore$ by addition $2=\frac{10}{3}x-\frac{8}{3}y$

i.e., $5x-4y=3$ or $y=\dfrac{5x-3}{4}$

Substituting in the first equation

$$\frac{x+\dfrac{5x-3}{4}}{1-x\dfrac{5x-3}{4}}=3 \text{ i.e., } \frac{9x-3}{4+3x-5x^2}=3.$$

This gives $x^2=1$, *i.e.*, $x=+1$ or -1

and correspondingly $y=\frac{1}{2}$ or -2.

Exercises

Solve the following equations :

1. $3x-2y=5$;
   $x^2+3xy+2y^2=35.$

2. $2x+\dfrac{1}{y}=5$;
   $y+\dfrac{4}{x}=3.$

3. $2x+3xy=8.$
   $y+2xy=6.$

4. $(x-a)(y-b)=ab$;
   $\dfrac{x}{a}=\dfrac{y}{b}.$

5. $x+y=7$;

$x^2+2y^2=33$.

6. $5x+12y=169$;

$\frac{5}{x}+\frac{12}{y}=2$.

7. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{7}{\sqrt{xy}}+1$; $\sqrt{xy}\,(x+y)=78$.

8. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}, =\frac{5}{2}x+y=10.$ (*P. U.* 1938)

9. $x+y=\frac{5}{6}$; $\frac{1}{x}-\frac{1}{y}=1$. (*P. U.* 1937)

10. $x+y+3\sqrt{x+y}=10$; $x^2+y^2=10$. (*P. U.* 1897)

11. $xy+x+y=27$. $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$. (*P. U.* 1942 Supp.)

12. $\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{5}{2}$; $x^2+y^2=90$ (*P. U.* 1902)

3·2. When terms containing x and y in both equations are homogenous, the proper method is to substitute $y=mx$ in both the equations.

Examples :—

(*i*) $3x^2-2y^2=10$, $x^2+xy+y^2=7$.

Substituting $y=mx$, in both the equations

$$x^2(3-2m^2)=10 \qquad \text{...(1)}$$

$$x^2(1-m+m^2)=7 \qquad \text{...(2)}$$

From (1) and (2) by division

$$\frac{3-2m^2}{1-m+m^2}=\frac{10}{7}\text{ , i.e., } 24m^2+10m-11=0$$

$$\therefore \qquad m=\frac{1}{2} \quad \text{or} \quad -\frac{11}{12}$$

If $m=\frac{1}{2}$, from (1) $x^2=4$, *i.e.*, $x=+2$ and -2 and correspondingly $y=1$ and -1

And if $m=-\frac{11}{12}$, from (1) $x^2=\frac{144}{19}$,

$$i.e.,\ x=\frac{12}{\sqrt{19}} \text{ or } \frac{-12}{\sqrt{19}}$$

and correspondingly, $y=\dfrac{-11}{\sqrt{19}}, \dfrac{11}{\sqrt{19}}$

(*ii*) $6x^2-5xy+y^2=0, \qquad 4x+xy+y^2=16.$

Putting $y=mx$, we get

$$x^2(6-5m+m^2)=0 \qquad \ldots(1)$$

and $$x^2(4+m+m^2)=16 \qquad \ldots(2)$$

From (1), $m=2$ or 3.

Putting $m=2$ in (2), $x^2=\dfrac{16}{10}$, $\therefore\ x=\dfrac{4}{\sqrt{10}}, \dfrac{-4}{\sqrt{10}}$

and correspondingly, $y=\dfrac{8}{\sqrt{10}}, \dfrac{-8}{\sqrt{10}}$.

Again putting $m=3$ in (2), $x^2=1$, $\therefore\ x=1, -1$

and correspondingly, $y=3, \ -3.$

Exercises

Solve the following equations :

1. $2x^2-3xy+y^2=0, \qquad x^2+5xy+6y^2=12.$
2. $x^2+3xy+2y^2=0, \qquad 2x^2-xy-y^2=9.$
3. $4x^2-xy+y^2=16, \qquad 3x^2-2xy+y^2=8.$ (*P. U.* 1910)
4. $x^2-y^2=15, \qquad 4x^2-3xy=18$ (*P. U.* 1912)
5. $x^2+xy=15, \qquad x^2-y^2=5$ (*P. U.* 1935)
6. $4x^2+7y^2=148, \qquad 12(x^2+y^2)=25xy$ (*P. U.* 1899)
7. $x^2+y^2=a^2, \qquad x+y=b.$
8. $x+y=2a, \qquad xy=k^2.$

3·3. If both the given equations are symmetrical in x and y (*i e.*, if the equations remain unchanged when x and y are interchanged), the proper course is to put $x=u+v$ and $y=u-v$.

Examples :

Ex. (*i*) $x^2+y^2=13, \qquad x+y=5.$

Put $x=u+v$, $y=u-v$; and we have from the second equation $u=\frac{5}{2}$. Hence $x=\frac{5}{2}+v$, $y=\frac{5}{2}-v$.

$\therefore$ from the first equation $(\frac{5}{2}+v)^2+(\frac{5}{2}-v)^2=13$

i.e., $2(\frac{25}{4}+v^2)=13$, giving $v=\frac{1}{2}$ or $-\frac{1}{2}$.

Taking $v=\frac{1}{2}$, we have $x=\frac{5}{2}+v=\frac{5}{2}+\frac{1}{2}=3$
and $y=\frac{5}{2}-v=\frac{5}{2}-\frac{1}{2}=2$.

Taking $v=-\frac{1}{2}$, we have $x=\frac{5}{2}+v=\frac{5}{2}-\frac{1}{2}=2$
and $y=\frac{5}{2}-v=\frac{5}{2}+\frac{1}{2}=3$.

Ex. 2. $x^2+y^2=5, \quad xy=2.$

Putting $x=u+v$, $y=u-v$ the equations become

$(u+v)^2-(u-v)^2=5$, *i.e.*, $u^2+v^2=\frac{5}{2}$ ...(*i*)

and $(u+v)(u-v)=2$, *i.e.*, $u^2-v^2=2$ ...(*ii*).

Form (*i*) and (*ii*) $u^2=\frac{9}{4}$, *i.e.*, $u=\frac{3}{2}$ and $-\frac{3}{2}$.

Form (*i*) if $u=\frac{3}{2}$, $v=\pm\frac{1}{2}$ and if $u=-\frac{3}{2}$, $v=\pm\frac{1}{2}$.

Hence the solution of (*i*) and (*ii*) are

$u=\frac{3}{2}$, $v=\frac{1}{2}$; $u=\frac{3}{2}$, $v=-\frac{1}{2}$;
$u=-\frac{3}{2}$, $v=\frac{1}{2}$ and $u=-\frac{3}{2}$, $v=-\frac{1}{2}$.

Taking these pairs of values of u and v in succession, we have $x=2, 1, -1, -2,$
and correspondingly $y=1, 2, -2, -1$.

Exercises

Solve the following :—

1. $x^4+y^4=626, \quad x+y=6.$
2. $x^4+y^4=97, \quad x-y=1.$
3. $x^2+y^2=41, \quad xy=20.$
4. $x^3+y^3=28xy, \quad x+y=12.$
5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x}{a}+\frac{y}{b}=1.$
6. $bx+ay=a^2+b^2, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{b^2}{a^2}+\frac{a^2}{b^2}.$
7. $\frac{1}{x}+\frac{1}{y}=3 ; \frac{1}{x^2}+\frac{1}{y^2}=5,$

8. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{13}{6}$; $x+y=13$.

9. $x^3+y^3=28$; $x+y=4$.

10. $x^3+y^3=a^3+1$; $xy=a$.

3·4. Equations containing more than one variable.

Examples :—

(*i*) Solve
$$x+y+z=0 \quad \ldots(1)$$
$$3x+2y+5z=0 \quad \ldots(2)$$
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=-\frac{7}{6} \quad \ldots(3)$$

From (1) and (2), by the rule of cross-multiplication,

$\frac{x}{5-2}=\frac{y}{3-5}=\frac{z}{2-3}$, *i.e.*, $\frac{x}{3}=\frac{y}{-2}=\frac{z}{-1}=k$, say

$\therefore\ x=3k, y=-2k, z=-k$.

$\therefore$ From (3) $\frac{1}{3k}-\frac{1}{2k}-\frac{1}{k}=-\frac{7}{6}$, whence $k=1$.

$\therefore\ x=3, y=-2, z=-1$.

(*ii*) $x(x+y+z)=6 \ldots\ldots\ldots(1)$

$y(x+y+z)=12 \ldots\ldots\ldots(2)$

$z(x+y+z)=18 \ldots\ldots\ldots(3)$

By addition $(x+y+z)^2=36$ *i. e.*, $x+y+z=\pm 6$.

Putting this in (1), (2) and (3), respectively, we have

$$x=\pm 1,\ y=\pm 2,\ z=\pm 3.$$

Hence $x=1,\quad y=2,\quad z=3$

and $\quad x=-1,\ y=-2,\ z=-3$,

are the two solutions.

(*iii*) $\frac{xy}{x+y}=1,\ \frac{yz}{y+z}=2,\ \frac{zx}{z+x}=3.$

These can be written as

$$\frac{1}{x}+\frac{1}{y}=1,\ \frac{1}{y}+\frac{1}{z}=\frac{1}{2},\ \frac{1}{z}+\frac{1}{x}=\frac{1}{3}.$$

$\therefore$ by addition $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=1+\frac{1}{2}+\frac{1}{3}=\frac{11}{6}$

i. e., $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{11}{12}$

But $\frac{1}{x}+\frac{1}{y}=1$, $\therefore$ $\frac{1}{z}=-\frac{1}{12}$ *i. e.*, $z=-12$.

Similarly $x=\frac{12}{5}$ and $y=\frac{12}{7}$.

Exercises

Solve the following equations :—

1. $xy=2$; $yz=8$; $zx=4$.
2. $3x+2y-7z=0$; $2x+3y-8z=0$; $x+y+z=4$.
3. $3x+2y-5z=0$; $4x+3y-2z=0$; $x^3+y^3+z^3=134$.
4. $2x+y+z=11$; $3x-y+5z=23$; $x^2+y^2+z^2+2yz=53$.
5. $x(y+z)=14$; $y(z+x)=18$; $z(x+y)=20$.
6. $yx+zx=9$; $zx+xy=5$; $xy+yz=8$.
7. $x^2yz=6$; $xy^2z=12$; $xyz^2=18$.
8. $xy+x+y=3$; $yz+y+z=7$; $zx+z+x=7$.
9. $\frac{xy}{x+y}=\frac{20}{9}$; $\frac{yz}{y+z}=\frac{30}{16}$; $\frac{zx}{x+y}=\frac{12}{5}$.
10. $x+y-z=14$; $y^2+z^2-x^2=46$; $yz=9$.
11. $zx+y=\frac{15}{2}x$; $xy+z=\frac{11}{3}x$; $x+y+z=10$.
12. $y+z : z+x : x+y=a : b : c$,
 $(y+z)^2+(z+x)^2+(x+y)^2=a^2+b^2+c^2$.

Revision Exercises

I

1. Solve $ax^2+bx+c=0$, and discuss the nature of its roots.

2. If α, β, be the roots of $ax^2+bx+c=0$, form an equation whose roots are α^2 and β^2.

3. Solve the equations :—

(*i*) $\frac{x+2}{x-2}+\frac{2x-3}{2(x-1)}=\frac{23}{6}$ (*P. U.* 1923, 1926)

(*ii*) $(2-x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}=1$

(*iii*) $\dfrac{2x-3}{\sqrt{x-2}+1}=2\sqrt{x-2}-1.$

4. Solve the following equations simultaneously :—

(*i*) $\dfrac{x+y}{1-xy}=3,\ \dfrac{x-y}{1+xy}=\dfrac{1}{3}.$

(*ii*) $\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{5}{2},\ x+y=10.$ (*P. U.* 1938)

(*iii*) $x^3+y^3=\frac{9}{2}xy,\ x+y=3.$

5. If x be real, prove that the expression $\dfrac{4x^2+2x-1}{x^2+6x+3}$ can have any real value.

II

1. Find the equation whose roots are the squares of the roots of $px^2=qx+r.$ (*P. U.* 1918)

2. Without solving $x^2+x+1=0$ actually, prove (*i*) the roots are reciprocal, (*ii*) one root is the square of the other. (*P. U.* 1922)

3. Prove that if the equation

$$(ay+\lambda)x^2+2(by+\mu)x+cy+\nu=0$$

has a root independent of y, then both the roots are rational, (a, b, c, λ, μ, ν being all real). (*P. U.* 1933)

4. Solve the following equations :—

(*i*) $\sqrt{x^2+x+3}+\sqrt{x^2-x+7}=6$ (*London* 1919)

(*ii*) $\dfrac{1}{(x-3)^4}+\dfrac{(x-3)^2}{x(x-6)}=\dfrac{7}{8}.$ (*Madras* 1895)

(*iii*) $(2+x)^{\frac{2}{3}}+(2-x)^{\frac{2}{3}}=\frac{5}{2}(4-x^2)^{\frac{1}{3}}.$ (*Madras* 1903)

5. Solve the following equations simultaneously :—

(*i*) $x^2-xy+y^2=7,\ x+y=5.$ (*P. U.* 1899)

(*ii*) $3x+4y=25,\ \dfrac{3}{x}+\dfrac{4}{y}=2.$ (*P. U.* 1937)

(*iii*) $\dfrac{x}{y}+\dfrac{y}{x}=18,\ x+y=12.$ (*C. U.* 1919)

III

1. Show that if the roots of $ax^2+bx+c=0$, are real, then those of

$$2(a+c)(ax^2+bx+c)+(b^2-4ac)(x^2+1)=0$$

are either equal or imaginary.

2. If the roots of $lx^2+nx+n=0$ be in the ratio $p:q$, prove that

$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0.$$

3. Find the value of $x^4+4x^3+6x^2+4x+9$, when

$$x=-1+\sqrt{-2}.$$

4. Solve the following equations :—

(*i*) $x^5+1=0$, (*ii*) $x^{-2}-10=3x^{-1}$. (*P. U.* 1936)

(*iii*) $6x^4-25x^3+12x^2+25x+6=0.$

5. Solve the following equations simultaneously :—

(*i*) $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{5}{2},\ x^2+y^2=90.$ (*P. U.* 1902)

(*ii*) $x^2+y(x+1)=137,\ y^2+x(y+1)=205.$ (*Madras* 1894)

(*iii*) $(x+2)(y+3)=24,\ xy=6.$ (*P. U.* 1900)

IV

1. If the roots of the equation

$$(c^2-ab)x^2-2(a-bc)x+(b^2-ac)=0,$$

are equal, prove that either $a+b+c=0$ or $a=b=c$.

2. Show that the equation $x^2+px+q=0$ will have a root common with the equation $x^2+qx+p=0$ if either $p=q$ or $p+q+1=0$.

3. If α, β be the roots of $ax^2+bx+c=0$, form an equation whose roots are $\dfrac{\alpha}{1+\beta}$ and $\dfrac{\beta}{1+\alpha}$.

4. Solve the following equations :—

(*i*) $\dfrac{1+x}{1+\sqrt{1+x}}+\dfrac{1-x}{1-\sqrt{1-x}}=1$

(*ii*) $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}=\dfrac{a}{b}+\dfrac{b}{a}$ (*P. U.* 1914)

(*iii*) $(12x-1)(16x-1)(4x-1)(3x-1)=5.$

5. Solve the following equations simultaneously :—

(*i*) $(x+y)^{\frac{2}{3}}+6(x-y)^{\frac{2}{3}}=5(x^2-y^2)^{\frac{1}{3}}$

$13x+18y=72$ (*Madras* 1900)

(*ii*) $3x^2-7y^2=5,\ 2xy-5y^2=1$

(*iii*) $\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=1,\ \sqrt{xy}\,(x+y)=78.$ (*P. U.* 1929)

V

1. If w denotes an imaginary cube root of unity, prove that

$(a+b+c)(a+w+wb+w^2c)(a+w^2b+wc)=a^3+b^3+c^3-3abc.$

2. Show that the necessary and sufficient condition that the roots of the equation $x^2+px+q=0$ be rational is $p=k+\dfrac{q}{k}$, where p, q, k are rational. (*P. U.* 1929)

3. If α, β be the roots of the equation $x^2+x+1=0$, from an equation whose roots are $\dfrac{\alpha}{\beta}$, $\dfrac{\beta}{\alpha}$ and account for the identity of the equation thus obtained with the original equation. (*P. U.* 1933)

4. Solve the following equations :—

(*i*) $\dfrac{(x-a)(x-b)}{x-a-b}=\dfrac{(x-c)(x-d)}{x-c-d}$ (*Math. Trip.*)

(*ii*) $\dfrac{x+\sqrt{x^2-2}}{x-\sqrt{x^2-2}}=\dfrac{x^2+1}{2}-x$ (*P. U.* 1904)

(*iii*) $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$

(*P. U.* 1919)

5. Solve the following equations simultaneously :—

(*i*) $x+xy=3,\quad y+xy=4$ (*Calcutta* 1921)

(*ii*) $x^2+y^2=7+xy,\ x^3+y^3=6xy-1.$ (*Madras* 1903)

(*iii*) $xy(x+y)=30,\ x^3+y^3=35.$

CHAPTER IV

Arithmetical Progression

4·1. Any succession of terms formed according to a definite law is called a series. Consider the following series : –

(1) $2, 5, 8, 11, \ldots\ldots$

(2) $6, 4, 2, 0, -2, -4, \ldots\ldots$

(3) $a, a+d, a+2d, \ldots\ldots$

(4) $1, 2, 4, 8, \ldots\ldots$

(5) $1, -3, 9, -27, 81, \ldots\ldots$

(6) $a, ar, ar^2, ar^3, \ldots\ldots$

(7) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\ldots$

(8) $\dfrac{1}{a}, \dfrac{1}{a+d}, \dfrac{1}{a+2d}, \ldots\ldots$

In (1), (2) and (3), the *difference* between any pair of successive terms is the same. Such a series is called an **Arithmetical Progression** (A. P.). The constant difference is called the **Common Difference.** The common difference for the series (1), (2) and (3) is respectively 3, -2 and d. Note that the common difference is obtained by subtracting any term from the one that *follows* it.

In (4), (5) and (6), the ratio of any term to the one that *precedes* it is the same. Such a series is called a Geometrical Progression (G. P.) and the constant ratio is called the *common ratio.* The common ratios for the series (4), (5) and (6) are 2, -3 and r respectively.

In (7) and (8), the reciprocals of the successive terms form a series in Arithmetical Progression. Such a series is called an Harmonic Progression. (H. P.)

4·2. *To find the nth term of an A. P., whose first term and common difference are given.*

Let a be the first term and d the common difference. Then if T_1, T_2, T_3 ....T_n denote the first, second, third...and nth terms respectively, we have

$$T_1=a=a+(1-1)d$$
$$T_2=a+d=a+(2-1)d$$
$$T_3=a+2d=a+(3-1)d$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

Generally, $T_n=a+(n-1)d$.

4·3. *To find the sum of the first n terms of an A. P.*

Let a, d, l denote the first term, the common difference and the last term of the given A. P. If S_n denote the sum of the first n terms, we have

$$S_n=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l$$

Also, writing the same series in the reverse order,

$$S_n=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a.$$

Adding $2S_n=(a+l)+(a+l)+(a+l)+$......upto n brackets.
$$=n(a+l).$$

So that $S_n=\frac{n}{2}(a+l)$ ... ... (i)

But $l=a+(n-1)d$,

$\therefore$ we also have

$$S_n=\frac{n}{2}[2a+(n-1)d]\ldots\ldots\ldots(2).$$

Similarly, since $a=l-(n-1)d$, we have

$$S_n=\frac{n}{2}[2l-(n-1)d]\ldots\ldots\ldots(3).$$

4·4. *Examples :—*

(i) If the first term and common difference of an A. P. be 3 and 2 respectively. Find the 20th term and the sum of the first 20 terms.

Sol. If T_{20} and S_{20} denote the 20th term and the sum of the first 20 terms respectively, we have

$$T_{20}=a+(20-1)d=3+(20-1)\times 2=41$$

and $S_{20}=\frac{20}{2}[2a+(20-1)d]=\frac{20}{2}[2\times 3+(20-1)\times 2]=440.$

(*ii*) If the tenth and the twelfth terms an A. P. be 20 and 24 respectively, find the 30th term and the sum of the first 30 terms.

Sol. Let a denote the first term and d the common difference of the given A. P. Then

$$T_{10}=a+9d=20$$

$$\text{and } T_{12}=a+11d=24.$$

These give $a=2$, $d=2$

$$\therefore \quad T_{30}=a+(30-1)d=2+(30-1)\times 2=60$$

$$\text{and} \quad S_{30}=\tfrac{30}{2}[2a+(30-1)d]=\tfrac{30}{2}[2\times 2+(30-1)\times 2]$$
$$=930.$$

(*iii*) The nth term of a series is given by the formula, $T_n=5n+4$. Prove that the series is in A. P.

Sol. Changing n to $n-1$, we have $T_{n-1}=5(n-1)+4$

$\therefore \quad T_n-T_{n-1}=[5n+4]-[5(n-1)+4]=5$, which is independent of n. Thus the difference between any two successive terms of the series is the same. Hence the series is an A. P. whose *c. d.* is 5.

(*iv*) Sum of the first n terms of a series is given by the formula $S_n=3n^2+2n+1$. Prove that the series is an A. P.

Sol. Changing n to $n-1$, we have

$$S_{n-1}=3(n-1)^2+2(n-1)+1$$

$$\therefore \quad S_n-S_{n-1}=[3n^2+2n+1]-[3(n-1)^2+2(n-1)+1]$$
$$=6n+2.$$

But S_n-S_{n-1} is the nth term. $\therefore \quad T_n=6n+2$

As in the last example, it can now be proved that the series is an A. P. whose common difference is 6.

(*v*) The sums of first n terms of two series in A. P. are in the ratio $2n+3 : 3n+4$. Find the ratio of their 10th terms.

Sol. Let a, β denote respectively the first term and the c. d. of the first series and let a', β' denote those of the second.

Then $\dfrac{\frac{n}{2}[2\alpha+(n-1)\beta]}{\frac{n}{2}[2\alpha'+(n-1)\beta']}=\dfrac{2n+3}{3n+4}$

i. e., $\dfrac{\alpha+\frac{n-1}{2}\beta}{\alpha'+\frac{n-1}{2}\beta'}=\dfrac{2n+3}{3n+4}$

Now the left-hand side represents the ratio of the 10th terms, provided $\frac{n-1}{2}=9$, *i. e.*, $n=19$.

$\therefore$ Putting $n=19$, the required ratio is

$$\frac{2\times19+3}{3\times19+4}=\frac{41}{61}.$$

Exercises

Find the general term (*i. e.* each term) of the following series :—

(*i*) $1+2+3+\ldots\ldots$ (*ii*) $1+\frac{3}{4}+\frac{1}{2}+\ldots\ldots$

(*iii*) $5+10+15+\ldots\ldots$ (*iv*) $100+95+90+\ldots\ldots$

(*v*) $7+13+19+\ldots\ldots$ (*vi*) $33+22+11+\ldots\ldots$

(*vii*) $-\frac{1}{3}, 0, \frac{1}{3}\ldots\ldots$ (*viii*) $-3, -7, -11\ldots\ldots$

(*ix*) $a-d, a-2d, a-3d\ldots\ldots$

(*x*) $a+2d, a+4d, a+6d\ldots\ldots$

2. Find the sum of the first n terms of the series given in the above question.

3. (*i*) Which term of the series (*iii*) in Q. 1 is 575

(*ii*) „ „ „ (*iv*) „ is 0

(*iii*) „ „ „ (*viii*) „ is -403.

4. (*i*) How many terms of the series (*iii*) in Q. 1 make the sum 1050.

(*ii*) How many terms of the series (*iv*) in Q. 1 make the sum 100.

(*iii*) How many terms of the series (*viii*) in Q. 1 make the sum 0.

Explain the double answer in each case.

5. Find the total number of terms in the series :—

(*i*) $5+7+9+......+205$

(*ii*) $8+4+0+......+(-200)$.

(*iii*) $\frac{1}{4}+\frac{1}{2}+\frac{3}{4}+......+50$.

6. Find the series whose nth term is

(*i*) $2n+1$; (*ii*) $11n+1$; (*iii*) $n+20$.

Also prove that these series are in A. P.

7. Find the series, the sum of whose first n terms is

(*i*) $\frac{n(n+1)}{2}$, (*ii*) an^2+bn+c. (*iii*) n^2-n+1.

8. Find the sum of all numbers which are less than 600 and divisible by 5.

9. Find the sum of all the integers from 1 to 300, excluding those that are multiples of 11.

10. Find the sum of all the integers from 1 to 300 excluding those that are multiples either of 11 or of 7.

11. The first term of an A. P. is unity and the ratio of the sum of first 7 terms to the sum of the next 7 terms is 13 : 41. Find the series.

12. The sum of n terms of an A. P. whose first term is 5 and c. d. is 36 is equal to the sum of $2n$ terms of another A. P. whose first term is 36 and c. d. is 5. Find n.

(*i*) $\frac{2x-y}{x+y}+\frac{3x-2y}{x+y}+\frac{4x-3y}{x+y}+......$ to $(x+y)$ terms

(*ii*) $\frac{1}{n}+\frac{n+1}{n}+\frac{2n+1}{n}+......$ to n terms

(*iii*) $\left(1-\frac{1}{n}\right)+\left(1-\frac{2}{n}\right)+\left(1-\frac{3}{n}\right)+......$ to n terms.

14. Obtain the sum of all numbers in the first 1000 integers which are neither divisible by 5 nor by 2.

(*P. U.* 1921 *Supp.*)

15. If T_p, T_q, and T_r denote the pth, qth, rth terms of an A. P., prove that

$$T_p(q-r)+T_q(r-p)+T_r(p-q)=0.$$

16. In an A. P., p times the pth term is equal to q times the qth term, prove that the $(p+q)$th term is zero.

17. There are two series in A. P. beginning with unity and the pth, qth, rth terms of the first A. P., are positive integer x, y, z respectively. Also the xth, yth and zth terms of the second A. P., are p, q, r respectively. Prove that the common differences of the two series are reciprocals of each other.

18. Find the number of terms of an A. P. in which the first term, the sum and the common difference are respectively $\frac{5}{7}$, 5, $-\frac{1}{20}$. (*P. U.* 1932)

19. The sum of the first p terms of a series in A. P. is zero. Prove that the sum of the next q term is

$$-\frac{a(p+q)q}{p-1}$$

20. The first, the rth and last term of an A. P. are a, b and l respectively. Prove that the sum of the series is

$$\frac{r-1}{2}\cdot\frac{l^2-a^2}{b-a}+\frac{l+a}{2}.$$

21. In an A. P. the sum up to p terms is a and the sum up to q terms is b, prove that the sum upto $p+q$ terms is

$$\frac{p+q}{2}\left(a+b+\frac{a-b}{p-q}\right).$$

22. The ratio of the sums up to n terms of two series in A. P. is

(*i*) $5n+4 : 6n+5$, (*ii*) $an+b : cn+d$.

Find the ratio of their 10th terms.

Arithmetic Means

4·5. Def. *A number A is said to be the* **Arithmetic Mean** *(A. M.) between a and b, if a, A, b are in A. P.*

Def. *Number $A_1, A_2 \ldots\ldots\ldots\ldots A_n$ are said to constitute n Arithmetic Means between a and b, if $a, A_1, A_2, \ldots\ldots\ldots A_n$, b form an A. P.*

(i) *To find the A. M. between two given numbers a and b.*

Let A be the required A. M. so that a, A, b are in A. P.

$$\therefore \quad b-A=A-a, \text{ whence } A=\frac{a+b}{2}.$$

Hence $\dfrac{a+b}{2}$ is the A. M. between a and b.

(ii) *To insert n A. M's between a and b.*

Let $A_1, A_2, \ldots\ldots, A_n$ be the required means.

Then $a, A_1, A_2, \ldots\ldots\ldots, A_n, b$ are in A. P.

This A. P. has $n+2$ terms in all. Therefore if d is the common difference, we have

$$b=T_{n+2}=a+(n+2-1)d=a+(n+1)d$$

$$\therefore \quad d=\frac{b-a}{n+1},$$

Now $\quad A_1=a+d=a+\dfrac{b-a}{n+1}=\dfrac{an+b}{n+1}$

$$A_2=a+2d=a+2.\frac{b-a}{n+1}=\frac{a(n-1)+2b}{n+1}$$

$$A_3=a+3d=a+3.\frac{b-a}{n+1}=\frac{b(n-2)+3b}{n+1}$$

and generally $A_r=a+rd=a+r\,\dfrac{b-a}{n+1}=\dfrac{a(n-r+1)+rb}{n+1}$

4·51. The word "Arithmetic Mean" is also used in a different sense. *The A. M. of n given numbers $a_1, a_2, \ldots\ldots, a_n$ is defined as the number $\dfrac{a_1+a_2+a_3+\ldots+a_n}{n}$*

The A. M. between any two given numbers is also the A. M. of any number of A. M.'s inserted between the given numbers.

Let a, b be the given numbers, A their single A. M. and $A_1, A_2, \ldots\ldots, A_n$, n Arithmetic Means inserted between them. Then by 4·5.

$$A_1+A_2+\ldots\ldots+A_n=(a+d)+(a+2d)+\ldots+(a+nd)$$

$$=na+d(1+2+\ldots+n)=na+\frac{n(n+1)}{2}d$$

$$=na+\frac{n(n+1)}{2}\cdot\frac{b-a}{n+1} \qquad \left(\because\ d=\frac{b-a}{n+1}\right)$$

$$=n\left(a+\frac{b-a}{2}\right)=n\,\frac{a+b}{2}=n\,\mathrm{A}.$$

$$\therefore \quad \mathrm{A}=\frac{A_1+A_2+\ldots A_n}{n},$$

which proves the statement.

4·52. *Examples* :—

(*i*) Insert 19 A. M.'s between 3 and 43.

Let $A_1, A_2, \ldots, A_{19}$ be the required means so that $3, A_1, A_2, \ldots, A_{19}, 43$ form an A. P. of $19+2=21$ terms. If d is the common difference of this A.P., we have

$$43=T_{21}=3+(21-1)d=3+20d, \text{ i.e., } d=2.$$

$\therefore\ A_r=3+rd=2+2r.$

Putting $r=1, 2, \ldots, 19$; we get the required means.

(*ii*) If n arithmetic means are inserted between 60 and 28 and the 5th A. M. is to $(n-5)$th as 5 : ,4 find n.

$$\text{Here } d=\frac{28-60}{n+1}=-\frac{32}{n+1}.$$

$$\therefore\ A_5=60-5\times\frac{32}{n+1}=\frac{60n-100}{n-1}$$

$$\text{and } A_{n-5}=60-(n-5)\times\frac{32}{n+1}=\frac{28n+220}{n+1}$$

$$\therefore\ A_5 : A_{n-5}=\frac{60n-100}{n+1}:\frac{28n+220}{n+1}$$

$$=15n-25 : 7n+55$$

$$\therefore \frac{15n-25}{7n+55}=\frac{5}{4}, \text{ i.e., } 60n-100=35n+275.$$

$$\therefore n=15.$$

Exercises

1. Insert one A. M. between

(*i*) 5 and 50 ; (*ii*) 25 and -25 ; (*iii*) $(a+b)^2$ and $(a-b)^2$

2. Insert

(*i*) 50 arithmetic means between 7 and 132.

(*ii*) 7 ,, ,, ,, $5\frac{1}{7}$ and $6\frac{2}{7}$.

(*iii*) n ,, ,, ,, 1 and n^2.

(*iv*) n ,, ,, ,, a and $a+n^3$.

3. Find n so that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the arithmetic mean between a and b. (*P. U.* 1922)

4. If a is the A. M. between a and b, prove that

(*i*) $(A-a)^2+(B-b)^2=\frac{1}{2}(a-b)^2$.

and (*ii*) $\dfrac{1}{\sqrt{A}+\sqrt{a}}+\dfrac{1}{\sqrt{A}+\sqrt{b}}=\dfrac{2}{\sqrt{a}+\sqrt{b}}$.

5. Show that the sum of n arithmetic means between a and b is to the sum of m arithmetic means between them as $n : m$.

4·6. The student must satisfy himself that if a given series is in A. P., then the series obtained by

(*i*) adding the same number from each term

(*ii*) subtracting the same number from each term

(*iii*) multiplying each term by the same number

(*iv*) dividing each term by the same number,

are also in A. P.

This principle is used in the following example.

4·61. Ex. 1. If a^2, b^2, c^2 are in A. P., prove that

$\dfrac{1}{b+c}$, $\dfrac{1}{c+a}$, $\dfrac{1}{a+b}$ are also in A. P. (*P. U.* 1900)

Sol. Since a^2, b^2, c^2 are in A. P.

$$a^2+bc+ca+ab,\ b^2+bc+ca+ab,\ c^2+bc+ca+ab$$

are also in A. P. (Adding the same number $bc+ca+ab$ to each term.)

i.e., $(c+a)(a+b)$, $(a+b)(b+c)$, $(b+c)(c+a)$ are also in A.P.

i.e., $$\frac{(c+a)(a+b)}{(a+b)(b+c)(c+a)},\ \frac{(a+b)(b+c)}{(a+b)(b+c)(c+a)},$$

$$\frac{(b+c)(c+a)}{(a+b)(b+c)(c+a)},$$

are also in A. P. [Dividing throughout by the same number]

$\therefore\ \dfrac{1}{b+c},\ \dfrac{1}{c+a},\ \dfrac{1}{a+b}$ are in A. P.

Ex. 2. The sum of three numbers in A. P. is 15 and the sum of their squares is 93. Find them.

Sol. Let the three numbers in A P. be

$$a-d,\ a,\ a+d.$$

$\therefore\ (a-d)+a+(a+d)=3a=15$ so that $a=5$.

The numbers are therefore $5-d$, 5, $5+d$,

where $\qquad (5-d)^2+5^2+(5+d)^2=93$

i.e., $\qquad 2d^2+75=93$ or $d^2=9$

$\therefore \qquad d=+3$ or -3.

Taking $d=+3$, we see that the numbers are 2, 5, 8.

Taking $d=-3$, we get the same numbers in the reverse order.

Note. When three numbers are given to be in A. P., it is useful to suppose them to be $a-d$, a, $a+d$.

Similarly if 4 numbers in A. P. are to be taken they should be

$$a-3d,\ a-d,\ a+d,\ a+3d.$$

Exercises

1. If a, b, c, be in A. P. prove that

(*i*) $\dfrac{1}{bc},\ \dfrac{1}{ca},\ \dfrac{1}{ab}$ are in A. P.

(ii) $\frac{a(b+c)}{bc}$, $\frac{b(c+a)}{ca}$, $\frac{c(a+b)}{ab}$ are in A. P.

(iii) $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ are in A. P.

(iv) $b+c$, $c+a$, $a+b$ are in A P.

(v) $(b+c)^2-a^2$, $(c+a)^2-b^2$, $(a+b)^2-c^2$ are in A. P.

2. If a^2, b^2, c^2, are in A P., then so are

(i) $\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$.

and (ii) $\frac{a}{b+c}$, $\frac{b}{c+a}$, $\frac{c}{a+b}$.

3. If $(b-c)^2$, $(c-a)^2$, $(a-b)^2$ are in A. P., then so are

$$\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}.$$

4. If $a(b+c)$, $b(c+a)$, $c(a+b)$ are in A. P., then so are

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}.$$

5. If $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ are in A. P. and $\frac{p-x}{px}=\frac{q-y}{qy}=\frac{r-z}{rz}$, then p, q, r are in A. P.

6. If a, b, c are in A. P., show that

(i) $a^2(b+c)+b^2(c+a)+c^2(a+b)=\frac{2}{9}(a+b+c)^3$

(ii) $a^3+4b^3+c^3=3b(a^2+c^2)$.

(**Hint.** Put $a=x-y$, $b=x$, $c=x+y$.)

7. If a, b, c, d be in A. P., prove that $\frac{a^2-d^2}{b^2-c^2}=3$.

8. Find the common difference of an A. P. whose first term is unity and sum to n terms n^2.

9. If in a series in A. P., $S_p=q$ and $S_q=p$, prove that $\frac{S_{p+q}}{p+q}+\frac{S_{p-q}}{p-q}=\frac{2q}{p}$ where S_n denotes the sum to n terms of the series.

10. The sum of x terms of an A. P. is equal to the sum of the next y terms and again equal to the sum of the next z terms ; prove that

$$(x+y)\left(\frac{1}{x}-\frac{1}{z}\right)=(x+z)\left(\frac{1}{x}-\frac{1}{y}\right)$$

11. If $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ be in A. P., so are

$(s-a)^2$, $(s-b)^2$, $(s-c)^2$, where $2s=a+b+c$.

12. Find 3 numbers in A. P. whose sum is 24 and whose product is 224.

13. Find three numbers in A. P. whose sum is 9 and the sum of whose squares is 35.

14. Find 4 numbers in A. P. whose sum is 24 and whose product is 945.

15. Four numbers in A. P. are such that the sum of their extremes is 10, while the product of their means is 35. Find them.

16. Divide 15 into three parts which are in A. P., and whose product is equal to 80.

17. Divide 12 into three parts which are in A. P. and the sum of whose products taken two by two is 39.

18. Divide 1 into three parts in A. P. the sum of whose squares is $\frac{31}{75}$.

19. A person saves each year Rs. 40 more than the previous years ; after 10 years his savings amount to Rs. 2000, excluding interest. What did he save in the first and the tenth year.

20. The interior angles of a convex polygon are in A. P. with common difference 10°, the least angle being 30°. Find the number of sides.

CHAPTER V.

Geometrical Progression.

5·1. **Def.** A series is said to be a Geometrical Progression (G. P.) if each term bears the same ratio to the one which immediately precedes it. This constant ratio is called the common ratio.

5.2. *To find the general term of a G. P.*

Let a be the first term and r the common ratio of a G. P. Then we have

$T_1=a=ar^{1-1}$, $T_2=ar=ar^{2-1}$, $T_3=ar^2=ar^{3-1}$...

Hence generally $T_n=ar^{n-1}$.

5·3. *To find the sum of the first n terms of a given G. P.*

Let S_n denote the sum of the first n terms of a series in G. P., whose first term is a and common ratio r. Then

$$S_n=a+ar+ar^2......+ar^{n-2}+ar^{n-1}$$
$$rS_n=\quad ar+ar^2+..............+ar^{n-1}+ar^n$$

Hence by subtraction

$$(1-r)S_n=a-ar^n=a(1-r^n)$$

$$\therefore \quad S_n=\frac{a(1-r^n)}{1-r} \text{ or}=\frac{a(r^n-1)}{r-1}.$$

5·4. **Examples** : –

(*i*) The second term of a G. P. is 10 and fourth term is 40. Find the sixth term.

Sol. Let a be the first term and r the common ratio of the given G. P.

Then $T_2=ar=10......(i)$

and $T_4=ar^3=40......(ii)$

From (*i*) and (*ii*) by division, $r^2=4$

$\therefore \quad T_6=ar^5=ar.r^4=10\times4\times4=160.$

(*ii*) If a, b, c are the pth, qth, rth term of a G. P., then $a^{q-r}\ b^{r-p}\ c^{p-q}=1.$

Sol. Let α, β denote the first term and the common ratio of the given G. P., so that

$$a=T_p=\alpha\beta^{p-1}\ldots\ldots(i)$$
$$b=T_q=\alpha\beta^{q-1}\ldots\ldots(ii)$$
$$c=T_r=\alpha\beta^{r-1}\ldots\ldots(ii)$$

From (i) and (ii), by division

$$\frac{a}{b}=\beta^{p-q} \text{ i. e., } \beta=\left(\frac{a}{b}\right)^{\frac{1}{p-q}} \ldots\ldots(iv)$$

Similarly from (ii) and (iii)

$$\frac{b}{c}=\beta^{q-r} \text{ i. e., } \beta=\left(\frac{b}{c}\right)^{\frac{1}{q-r}} \ldots\ldots(v)$$

From (iv) and (v),

$$\left(\frac{a}{b}\right)^{\frac{1}{p-q}}=\left(\frac{b}{c}\right)^{\frac{1}{q-r}}$$

$$\therefore \left(\frac{a}{b}\right)^{q-r}=\left(\frac{b}{c}\right)^{p-q}$$

or $a^{q-r}.c^{p-q}=b^{p-q}.b^{q-r}=b^{p-r}=\dfrac{1}{b^{r-p}}$

$$\therefore a^{q-r}.\,b^{r-p}.\,c^{p-q}=1.$$

(iii) Find the sum to n terms of the series

$$2+3+\tfrac{9}{2}+\ldots\ldots$$

Here $a=2$, $r=\frac{3}{2}$, $\therefore$ $S_n=\dfrac{a(r^n-1)}{r-1}=\dfrac{2[(\frac{3}{2})^n-1]}{\frac{3}{2}-1}$

$$=2^2[(\tfrac{3}{2})^n-1]=\frac{(3^n-2^n)}{2^{n-2}}.$$

(iv) Find the sum of the first n terms of the series

$$7+77+777+\ldots\ldots$$

Sol. Here $S=7+77+777+\ldots\ldots$ upon n terms

$$=7[1+11+111+\ldots\ldots]$$

$$=\tfrac{7}{9}[9+99+999+\ldots\ldots]$$

$$=\tfrac{7}{9}[(10-1)+(10^2-1)+(10^3-1)+\ldots\ldots]$$

$$=\tfrac{7}{9}[(10+10^2+10^3+\ldots\ldots \text{to } n \text{ terms})-n]$$

$$=\tfrac{7}{9}\left[10.\frac{10^n-1}{10-1}-n\right]=\frac{7(10^{n+1}-9n-10)}{81}.$$

(*v*) Sum to n terms the series

$$\cdot 7+\cdot 77+\cdot 777+\ldots\ldots$$

Sol. Here $S=\cdot 7+\cdot 77+\cdot 777+\ldots\ldots$ to n terms

$$=\tfrac{7}{9}[\cdot 9+\cdot 99+\cdot 999+\ldots\ldots]$$

$$=\tfrac{7}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+.\right]$$

$$=\tfrac{7}{9}\left[n-\frac{1}{10}\cdot\frac{1-\frac{1}{10^n}}{1-\frac{1}{10}}\right.$$

$$=\tfrac{7}{9}\left(n-\tfrac{1}{9}+\frac{1}{9.10^n}\right).$$

(*vi*) The sum of three numbers in G. P. is 14 and their product is 64. Find them.

Sol. Let the three numbers in G. P. be $\frac{a}{r}$, a, ar.

$\therefore$ their product$=\frac{a}{r}.a.ar=a^3=64$, $\therefore$ $a=4$

Hence the numbers are $\frac{4}{r}$, 4, $4r$

$\therefore$ their sum$=\frac{4}{r}+4+4r=14.$

$\therefore$ $2r^2-5r+2=0$, giving $r=2$ or $\frac{1}{2}$.

Taking $r=2$, we find that the numbers are 2, 4, 8.

Taking $r=\frac{1}{2}$, we get the same numbers in the reverse order.

Examples

1. Find the nth term and sum of the first n terms of the following Geometrical progressions :—

(*i*) $1, 3, 9, \ldots\ldots$

(*ii*) $2, -\frac{4}{9}, \frac{8}{81}, \ldots\ldots$

(*iii*) $(\sqrt{2}+1), 1, \sqrt{2}-1, \ldots\ldots$

(*iv*) $(\sqrt{a}+1), 1, (\sqrt{a}-1), \ldots\ldots$

2. Prove that in a G. P., the product of any two terms equidistant from the beginning and the end is independent of common ratio.

3. If a, b, c be the pth, qth and rth term of a G. P., respectively, prove that $a^{q-r}.b^{r-p}.c^{p-q}=1$.

4. Find the nth term of a G. P. whose pth term is a and qth term is b.

5. If in a G. P. the xth power of the xth term is equal to the yth power of the yth term, prove that the $(x+y)$th term must be unity.

6. Show that the Arithmetic means between x and y, $\sqrt{\frac{x}{y}}$ and $\sqrt{\frac{y}{x}}$, $\frac{1}{x}$ and $\frac{1}{y}$ are in G. P.

7. Find a G P. of 4 terms such that the third term is greater by 2 than the sum of the first and second ; and the fourth terms is greater by 4 than the sum of the second and third. (*P. U.* 1901)

8. Sum to n terms the following series :—

(*i*) $(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots\ldots$

(*ii*) $x(x+y)+x^2(x^2+y^2)+x^3(x^3+y^3)+\ldots\ldots$

(*iii*) $1+x(1+x)+x^2(1+x+x^2)+\ldots\ldots$

(*iv*) $(a+b)+(a^2+2b)+(a^3+3b)+\ldots\ldots$ (*P. U.* 1936)

(*v*) $(3x+x^2)+(4x-x^3)+(5x+x^4)+\ldots\ldots$

(*vi*) $1+(1+\frac{1}{2})+(1+\frac{1}{2}+\frac{1}{4})+\ldots\ldots$ (*Allahabad* 1212)

(*vii*) $1+b+bc+b^2c+b^2c^2+b^3c^2+b^3c^3+\ldots\ldots$

(*viii*) $\left(2^n+\frac{1}{2^n}\right)+\left(2^{n-1}+\frac{1}{2^{n-1}}\right)+\ldots\ldots+(2+\frac{1}{2})+1.$

9. Sum to n terms the following series :—

(*i*) $3+33+333+\ldots\ldots$

(*ii*) $4+44+444+\ldots\ldots$

(*iii*) $\cdot3+\cdot33+\cdot333+\ldots\ldots$

(*iv*) $\cdot7+\cdot77+\cdot777+\ldots\ldots$

(*v*) $\frac{1}{3}+\frac{2}{3^2}+\frac{1}{3^3}+\frac{2}{3^4}+\ldots\ldots$

(*vi*) $\frac{1}{10}+\frac{2}{10^2}+\frac{3}{10^3}+\frac{1}{10^4}+\frac{2}{10^5}+\frac{3}{10^6}+\ldots\ldots$

10. Show that if pth, qth, rth terms of a G. P. are in G. P., then p, q, r are in A. P.

5·5. Infinite Geometric Series.

To show that, if r is numerically less than unity, $r^n \to 0$ as $n \to \infty$.

To be specific let us take $r=\frac{1}{2}$

We have $r^2=r.r=\frac{1}{2}r<r$

$r^3=r.r^2=\frac{1}{2}.r^2<r^2$

$r^4=r.r^3=\frac{1}{2}.r^3<r^3$ and so on

Generally $r^n=r.r^{n-1}=\frac{1}{2}.r^{n-1}<r^{n-1}$.

Thus as n continually increases, r^n decreases continually and there is no limit to its decrease *i. e.*, it can be made as small as we please. Hence the result.

5·51. *To find the sum of an infinite G. P. whose common ratio is numerically less than unity.*

Let the G. P. be

$a+ar+ar^2+\ldots\ldots ar^n+\ldots\ldots$ to infinity $(-1<r<1)$

By Art. 5·4, if S_n denotes the sum to n terms and S denotes the sum to infinity we have,

$$S_n=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{a}{1-r}.r^n$$

From Art. 5·5, since r is numerically less than unity $r^n \to 0$ as $n \to \infty$.

$$\therefore\ S_n \to \frac{a}{1-r} \text{ or } S=\frac{a}{1-r}.$$

Note. The sum to infinity of a G. P. has a meaning only if r is numerically <1.

Ex. 1. (*i*) In an infinite G. P., prove that each term bears a constant ratio to the sum of all the terms that follows it.

Let $a+ar+ar^2+\ldots\ldots+ar^{n-1}+ar^n+ar^{n+1}+\ldots\ldots\ldots$ be the infinite G. P., where r is numerically less than 1.

Here $T_n=ar^{n-1}$, and if S denotes the sum of all the terms that follow T_n, we have

$$S=ar^n+ar^{n+1}+ar^{n+2}+\ldots\ldots \text{to infinity}$$

$$=\frac{ar^n}{1-r}$$

$$\therefore\quad \frac{T_n}{S}=\frac{ar^{n-1}}{(ar^n/1-r)}=\frac{1-r}{r}, \text{ which is independent of } n.$$

(*ii*) The sum to infinity of a G. P. is $\frac{4}{3}$ and each term is 3 times the sum of all the terms that follow it. Find the G. P.

Let a be the first term and r the common ratio of the G. P., then $\frac{a}{1-r}=\frac{4}{3}\ldots\ldots(i)$

Also from the above example

$$\frac{1-r}{r}=3 \text{ i. e., } 1-r=3r$$

Hence $r=\frac{1}{4}$ and from (*i*) $a=1$

$\therefore$ The G. P. is $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}\ldots\ldots$

5·6. Def. 1. A number G is said to be a Geometric Mean (G. M.) between two numbers a and b, if a, G, b are in G. P.

Def. 2. Numbers $G_1, G_2,\ldots\ldots G_n$ are said to be n Geometric means between a and b, if

$$a, G_1, G_2,\ldots\ldots G_n, b \text{ are in G. P.}$$

5·61. *To find the G. M. between a and b.*

Let G be the required G. M., so that a, G, b are in G. P.

$$\therefore \quad \frac{G}{a}=\frac{b}{G}, \text{ or } G^2=ab,$$

$$\therefore \quad G=\sqrt{ab}.$$

5·62. *To insert n G. M.'s between a and b.*

Det $G_1, G_2, G_3, \ldots\ldots, G_n$ be the required Geometric means, so that

$$a, G_1, G_2, \ldots\ldots G_n, b \text{ are in G. P.}$$

Let r be the common ratio. Then

$$b=T_{n+2}=ar^{n+1}, \quad \therefore \quad r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Hence $G_1=ar=a.\left(\frac{b}{a}\right)^{\frac{1}{n+1}}=a^{\frac{n}{n+1}}.b^{\frac{1}{n+1}}$

$$G_2=ar^2=a.\left(\frac{b}{a}\right)^{\frac{2}{n+1}}=a^{\frac{n-1}{n+1}}.b^{\frac{2}{n+1}}$$

$$G_3=ar^3=a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}=a^{\frac{n-2}{n+1}}.b^{\frac{3}{n+1}}$$

... ...
... ...
... ...

Finally $G_n=ar^n=a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}=a^{\frac{1}{n+1}}.b^{\frac{n}{n+1}}$.

5·7. Examples :—

(*i*) Prove that the A. M. between two positive numbers is greater than their G. M.

Let A and G denote the Arithmetic and Geometric means between a and b.

Then $\quad A=\dfrac{a+b}{2}$ and $G=\sqrt{ab}$

Now $A-G=\dfrac{a+b}{2}-\sqrt{ab}=\dfrac{(\sqrt{a})^2+(\sqrt{b})^2-2\sqrt{a}\sqrt{b}}{2}$

$$=\frac{(\sqrt{a}-\sqrt{b})^2}{2} \text{ which is always positive.}$$

Hence $A>G$.

(*b*) Prove that the *n*th power of the G. M. between two numbers is equal to the product of *n* G. M.'s inserted between the numbers.

Let a, b be the two numbers, G. their G. M. and G_1, G_2, ...G_n the n. G. M 's inserted between a and b.

Then $G_1\,G_2.G.........G_n$

$$=\left\{a^{\frac{n}{n+1}}.\,b^{\frac{1}{n+1}}\right\}\left\{a^{\frac{n-1}{n+1}}.\,b^{\frac{2}{n+1}}\right\}\left\{a^{\frac{n-2}{n+1}}.\,b^{\frac{3}{n+1}}\right\}......\left\{a^{\frac{1}{n+1}}.\,b^{\frac{n}{n+1}}\right\}$$

$$=a^{\frac{n}{n+1}+\frac{n-1}{n+1}+.........+\frac{1}{n+1}}\times b^{\frac{1}{n+1}+\frac{2}{n+1}+......+\frac{n}{n+1}}$$

$$=(ab)^{\frac{n}{2}}=(\sqrt{ab})^n=G^n.$$

Exercises

Sum to infinity the following series .—

1. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+......$
2. $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+......$
3. $\frac{1}{5}+\frac{2}{5^2}+\frac{1}{5^3}+\frac{2}{5^4}+......$
4. $\frac{1}{7}+\frac{3}{7^2}+\frac{5}{7^3}+\frac{1}{7^4}+\frac{3}{7^5}+\frac{5}{7^6}+\frac{1}{7^7}+......$

5. $1+x+x^2+x^3+\ldots\ldots(x<1)$.

6. $\sqrt{3}+\dfrac{1}{\sqrt{3}}+\dfrac{1}{3\sqrt{3}}+\ldots\ldots$ (*P. U.* 1917)

7. $3+\cdot 3+\cdot 03+\ldots\ldots$

8. $50-40+32-25\cdot 6+20\cdot 48\ldots\ldots$ (*P. U.* 1937)

9. $\frac{5}{2}-1+\frac{2}{5}\ldots\ldots$ (*P. U.* 1921)

10. $1+(1+b)r+(1+b+b^2\,r^2+\ldots\ldots$ (*Allahabad* 1927)

Express the following recurring decimals as infinite Geometric series and hence reduce them to vulgar fractions :—

11. $\cdot\dot{5}$. 12. $\cdot 3\dot{7}$. 13. $2\cdot 34\dot{1}\dot{7}$.

14. $0\cdot 2\dot{3}\dot{4}$ (*P. U.* 1934)

15. The sum to infinity of a G. P. is 1 and the sum of the square of the terms is $\frac{4}{3}$. Find the G. P.

16. The first two terms of an infinite G. P. are together equal to 1 and every term is twice the sum of all the terms that follow it. Find out the series and the sum to infinity. (*P. U.*)

17. In an infinite G. P., each term is 3 times the sum of all the terms which follow it. The sum of the first two terms is 15. Find the sum to infinity. (*P. U.* 1922)

18. Insert (*i*) 4 Geometric means between 5 and 1215. (*P. U.*)

(*ii*) 6 „ „ „ 56 and $-\frac{7}{18}$. *P. U.* 1913)

19. The sum of two quantities is n times their Geometric means, show that the quantities are in the ratio

$$(n+\sqrt{n^2-4}):(n-\sqrt{n^2-4}).$$ (*P. U.* 1921)

20. If a, b, c are in G. P., and x, y are respectively the Arithmetic means between a, b, and b, c; prove that

(*i*) $\dfrac{a}{x}+\dfrac{c}{y}=2$; (*ii*) $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{2}{b}$ (*P U.*1892)

21. The difference between two numbers is 48 and their A. M. exceeds their G. M. by 18, determine the numbers. (*P. U.* 1894)

22. The Arithmetic mean of two integral numbers exceed their Geometric mean by 2 and the ratio of the numbers is 4, find the numbers. (*P. U.* 1910)

23. Find the value of n, if $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the G. M. between a and b.

24. Find the sum of n Geometric means inserted between a and b.

25. If $g_1, g_2, \ldots\ldots g_m$ be m Geometric means and $G_1, G_2, \ldots\ldots G_n$ be n Geometric means between a ond b, prove that

$$(g_1g_2\ldots\ldots g_m)^n=(G_1G_2\ldots\ldots G_n)^m.$$

58. Miscellaneous Examples.

Ex. 1. Four numbers are in A. P. and if they are incrcased by 2, 5, 20, 71 respectively, they form a G. P. Find them.

Sol. Let the resulting G. P. be a, ar, ar^2, ar^3

Then $a-2, ar-5, ar^2-20, ar^3-71$ are in A. P.

$$\therefore\ (ar-5)-(a-2)=(ar^2-20)-(ar-5)$$
$$=(ar^3-71)-(ar^2-20).$$

These may be written as

$$ar^2-2ar+a-12=0\ldots\ldots(i)$$

and $$ar^3-2ar^2+ar-36=0\ldots\ldots(ii)$$

Now (ii) can be written as

$$r(ar^2-2ar+a)-36=0$$

or $$r\times12-36=0,\ \therefore\ r=3.$$

Hence from (i) $a=3$

$\therefore$ The four numbers in A. P. are 1, 4, 7, 10.

Ex. 2. If a, b, c, d are in G. P., prove that

$a+b, b+c, c+d$ are also in G. P.

Sol. Let $\frac{b}{a}=\frac{c}{b}=\frac{d}{c}=r,$

$$\therefore\ b=ar,\ c=br=ar^2,\ d=cr=ar^3$$

$$\therefore\ \frac{b+c}{a+b}=\frac{ar+ar^2}{a+ar}=r$$

and $$\frac{c+d}{b+c}=\frac{ar^2+ar^3}{ar+ar^2}=r$$

$\therefore\ \frac{b+c}{a+b}=\frac{c+d}{b+c}$ so that $a+b, b+c, c+d$ are in G. P.

Exercises

1. If the numbers 1, 1, 3, 9 be added to four numbers in A P., they form a G. P. Find the first term and c d. of this A. P. (*P. U.* 1907)

2. Determine four numbers in G. P., such that their sum is 15 and the sum of their squares is 85. (*P. U.* 1933)

3. If a, b, c are in A. P. and x, y, z are in G. P., prove that $x^b y^c z^a = x^c y^a z^b$. (*P. U.* 1935)

4. The sum of four numbers in G. P. is 60, and the A. M. between the first and last is 18. Find them. (*P. U.* 1940)

5. Three quantities in G. P. have a sum $\frac{7}{8}$ and the sum of their squares is $\frac{133}{88}$. Find them. (*P. U.*)

6. If a, b, c are in A. P. and a, b, d are in G. P. show that $a, a-b, d-c$ are in G. P. (*P. U.* 1909)

7. From three numbers in G. P., three others in G. P are substracted and the resulting numbers are also in G. P. Show that the three Geometric progressions have the same common ratio. (*Allahabad* 1930)

8. If a, b, c, d be in G. P., prove that

( ) $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

(ii) $(b+c)(b+d)=(c+a)(c+d)$.

(iii) $(a^2+b^2)(c^2+d^2)=(b^2+c^2)^2$.

(iv) $(a-b)^2(c-d)^2=(b-c)^4$.

(v) $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)=\frac{(a+b+c+d)^2}{abcd}$.

9. If $x=1+a+a^2+$......ad. inf.

and $y=1+b+b^2+$......ad. inf.

prove that $1+ab+a^2b^2+$......ad. inf. $=\frac{xy}{x+y-1}$

10. Take all integers from 2 to infinity and raise each to all negative integral powers from 2 to infinity ; show that the sum of all these quantities is equal to unity. (*P. U.* 1924)

CHAPTER VI

Harmonical Progression

6·1. Def. A series is said to be in Harmonical Progression (H. P) if the reciprocals of its terms form an A. P. Every H. P. is therfore of the form

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots\ldots, \frac{1}{a+\overline{n-1}d}$$

Theorem. *If a, b, c, are in H. P., then*

$$a-b : b-c = a : c,$$

and conversely.

Proof. Since a, b, c are in H. P.

$$\therefore \quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in A. P.}$$

$$\therefore \quad \frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$$

$$\therefore \quad \frac{a-b}{ab}=\frac{b-c}{bc}$$

$$\text{or} \quad \frac{a-b}{b-c}=\frac{ab}{bc}=\frac{a}{c}$$

The converse follows easily.

In view of this theorem an H. P. may also be defined as follows :-

A number of quantities a, b, c, d, e,...... are said to form an H. P., if

$$\frac{a-b}{b-c}=\frac{a}{c}; \frac{b-c}{c-d}=\frac{b}{d}, \frac{c-d}{d-e}=\frac{c}{e} \text{ and so on.}$$

6·2. *To find the nth term of a series in H. P., whose first two terms are given.*

Let a, b be the first and second terms of the given H. P., so that $\frac{1}{a}$, $\frac{1}{b}$ are the first two terms of the corresponding

A. P. If d is the common difference of this A. P., we have

$$d=\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}.$$

$$\therefore\ n\text{th term of the A. P.}=\frac{1}{a}+\overline{n-1}\,d$$

$$=\frac{1}{a}+(n-1)\cdot\frac{a-b}{ab}$$

$$=\frac{a(n-1)-b(n-2)}{ab}$$

$$\therefore\ n\text{th term of the H. P.}=\frac{ab}{(n-1)a-(n-2)b}.$$

N. B. There is no formula for finding the sum of n terms of an H. P.

6·3. Def. If a, b, c are in H. P., then b is called the Harmonic Mean (H. M.) between a and b.

To find the Harmonic Mean between a and b.

Let H be the required H. M. so that

$$a,\ \text{H},\ b \text{ are in H. P.}$$

$$\therefore\ \frac{1}{a},\ \frac{1}{\text{H}},\ \frac{1}{b} \text{ are in H. P.}$$

$$\therefore\ \frac{2}{\text{H}}=\frac{1}{a}+\frac{1}{b}$$

$$\text{or } \text{H}=\frac{2ab}{a+b}.$$

6·31. Def. Numbers H_1, H_2,......, H_n are said to be n Harmonic Means between a and b, if a, H_1, H_2,......, H_n, b are in H. P.

To insert n Harmonic Means between a and b.

Let H_1, H_2, ..., H_n be the required means, so that

$$a,\ \text{H}_1,\ \text{H}_2,\ldots\ldots,\ \text{H}_n,\ b \text{ are in H. P.}$$

$$\therefore\ \frac{1}{a},\ \frac{1}{\text{H}_1},\ \frac{1}{\text{H}_2},\ \ldots\ldots,\ \frac{1}{\text{H}_n},\ \frac{1}{b} \text{ are in A. P.}$$

Let d be the common difference of this A. P.

$$\therefore \quad \frac{1}{b}=T_{n+2}=\frac{1}{a}+(\overline{n+2}-1)d$$

$$\therefore \quad d=\left(\frac{1}{b}-\frac{1}{a}\right)\Big/ n+1=\frac{a-b}{(n+1)ab}$$

$$\therefore \quad \frac{1}{H_1}=\frac{1}{a}+d=\frac{1}{a}+\frac{a-b}{(n+1)ab}$$

$$=\frac{a+nb}{(n+1)ab}$$

$$\therefore H_1=\frac{(n+1)ab}{a+nb}.$$

Similarly $\quad H_2=\dfrac{(n+1)ab}{2a+(n-1)b}$.

$H_3=\dfrac{(n+1)ab}{3a+(n-2)b}$ and so on.

Lastly $\quad H_n=\dfrac{(n+1)ab}{na+b}$.

6·32. **Theorem** *If A, G, H, denote respectively the Arithmetic, Geometric and Harmonic Means between two positive quantities a and b ; then*

(b) A>G>H ; (ii) A, G, H are in G. P.

Proof. We have $A=\dfrac{a+b}{2}$, $G=\sqrt{ab}$, $H=\dfrac{2ab}{a+b}$.

$\therefore$ $A-G=\dfrac{a+b}{2}-\sqrt{ab}=\dfrac{(\sqrt{a}-\sqrt{b})^2}{2}$ which is positive.

Hence A>G.

Similarly $G-H=\sqrt{ab}-\dfrac{2ab}{a+b}$

$$=\frac{\sqrt{ab}}{a+b}((\sqrt{a})^2+(\sqrt{b})^2-2\sqrt{ab})$$

$$=\frac{\sqrt{ab}}{a+b}(\sqrt{a}-\sqrt{b})^2,$$

which is positive. Hence G>H.

$\therefore \quad$ A>G>H.

Also A.H. $=\frac{a+b}{2}\times\frac{2ab}{a+b}=ab=G^2$,

$\therefore\ \frac{G}{A}=\frac{H}{G}$ $\qquad\therefore$ A, G, H are in G. P.

6·4. Illustrative examples.

1. The sum of three numbers in H. P. is 13 and the sum of their reciprocals is $\frac{13}{4}$. Find them.

Let the three numbers in H. P. be

$$\frac{1}{a-d},\ \frac{1}{a},\ \frac{1}{a+d}$$

so that $\frac{1}{a-d}+\frac{1}{a}+\frac{1}{a+d}=13$ ......(1)

and $(a-d)+a+(a+d)=\frac{3}{4}$, *i.e.*, $a=\frac{1}{4}$.

$$\therefore\quad \frac{1}{\frac{1}{4}-d}+\frac{1}{\frac{1}{4}}+\frac{1}{\frac{1}{4}+d}=13$$

i.e., $\frac{4}{1-4d}+\frac{4}{1+4d}=13-4=9$

or $\quad 1-16d^2=\frac{8}{9}$, $\therefore d=\pm\frac{1}{12}$.

The numbers are found to be 3, 4, 6.

2. The pth term of an H. P. is q and the qth term is p; prove that the (pq)th term is 1.

The pth and qth terms of the corresponding A. P. are respectively $1/q$ and $1/p$. Let a be the first term and d the common difference of this A. P., so that

$$\frac{1}{q}=a+(p-1)d \text{ and } \frac{1}{p}=a+(q-1)d,$$

From these, we have

$$a=\frac{1}{pq},\ d=\frac{1}{pq}$$

$\therefore$ (pq) the term of the A. P. $=a+(pq-1)d$

$$=\frac{1}{pq}+(pq-1)\frac{1}{pq}=1.$$

Hence the (pq) the term of the H. P. $=1$.

3. If $a^x = b^y = c^z$, and a, b, c are in G. P. show that x, y, z are in H. P.

We have $a=b^{y/x}$, $c=b^{y/z}$,

Also since $a, b, c,$ are in G. P., $ac=b^2$

$\therefore \quad b^{y/x} \cdot b^{y/z}=b^2$

or $\quad \dfrac{y}{x}+\dfrac{y}{z}=2$ *i. e.*, $\dfrac{1}{x}+\dfrac{1}{z} = \dfrac{2}{y}$

$\therefore \quad \dfrac{1}{x}, \quad \dfrac{1}{y}, \quad \dfrac{1}{z}$ are in A.P.

$\therefore \quad x, y, z$ are in H. P.

4. If $a, b, c,$ are in A. P. $c, a, b,$ are in G. P. prove that b, c, a are in H. P.

Let $b=a+d$, $c=a+2d$.

Now $bc=a^2$, $\because$ c, a, b are in G. P.

$\therefore \quad (a+d)(a+2d)=a^2$, giving $d= -\dfrac{3a}{2}$

Now b, c, a are in H. P.

if $a+d$, $a+2d$, a are in H. P.

i. e., if $a-\dfrac{3a}{2}$, $a-3a$, a are in H.P.

i. e., if $-\dfrac{a}{2}$, $-2a$, a are in H. P.

i. e., if $-\dfrac{2}{a}, -\dfrac{1}{2a}, \quad \dfrac{1}{a}$ are in A. P.

which is true $\therefore$ b, c, a are in H. P.

5. Two arithmetic means x_1, x_2, two geometric means y_1, y_2 and two harmonic means z_1, z_2 are inserted between a and b, show that $y_1y_2 : z_1z_2=x_1+x_2 : z_1+z_2$.

Since a, x_1, x_2, b are in A. P. $\therefore x_1+x_2=a+b$.

Also a, y_1, y_2, b are in G. P., $\therefore y_1y_2=ab$.

Lastly $a, z_1, z_2, b,$ are in H. P.

i. e. $\frac{1}{a}, \frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{b}$ are in A. P.

$$\therefore \quad \frac{1}{z_1}+\frac{1}{z_2}=\frac{1}{a}+\frac{1}{b}$$

or $$\frac{z_1+z_2}{z_1z_2}=\frac{a+b}{ab}=\frac{x_1+x_2}{y_1y_2}$$

$\therefore y_1y_2 : z_1z_2=x_1+x_2 : z_1+z_2.$

6. If x_1, x_2, x_3, x_4, are in H. P. Prove that

$$x_1x_2+x_2x_3+x_3x_4=3x_1\, x_4$$

If d be the common difference of the corresponding A. P., we have

$$\frac{1}{x_2}-\frac{1}{x_1}=d, \ i.\ e.,\ x_1\, x_2=\frac{x_1-x_2}{d}$$

$$\frac{1}{x_3}-\frac{1}{x_2}=d, \ i.\ e.,\ x_2\, x_3=\frac{x_2-x_3}{d}$$

$$\frac{1}{x_4}-\frac{1}{x_3}=d, \ i.\ e.,\ x_3\, x_4=\frac{x_3-x_4}{d}$$

Also $\frac{1}{x_4}-\frac{1}{x_1}=3d.\ i.\ e.,\ x_1\, x_4=\frac{x_1-x_4}{3d}$

Thus $$x_1x_2+x_2x_3+x_3x_4=\frac{(x_1-x_2)+(x_2-x_3)+(x_3-x_4)}{d}$$

$$=\frac{x_1-x_4}{d}=3\cdot\frac{x_1-x_4}{3d}$$

$$=3x_1x_4.$$

Exercises

Find the nth term of the following series :—

1. $4+4\frac{2}{7}+4\frac{8}{13}+\ldots\ldots$ (*P. U.* 1916)

2. $5+7+\frac{35}{3}+\ldots\ldots$

3. $8+4+\frac{8}{3}+\ldots\ldots$

4. $\frac{8}{4}+\frac{10}{9}+\frac{10}{13}+\ldots\ldots$

Continue the following Harmonic progressions to 5 terms :—

5. $\frac{1}{3}+\frac{3}{10}\ldots\ldots$

6. $\frac{1}{3}+\frac{2}{7}+\ldots\ldots$

7. The first and third terms of an H. P. are respectively $\frac{1}{2}$ and $\frac{1}{4}$. Find the 10th term.

8. Insert 15 harmonic means between 1 and 17.

9. Insert 3 harmonic means between 3 and 19.

10. The geometric and harmonic means between two numbers are 6 and $\frac{18}{5}$ respectively. Find them.

11. The difference between two numbers is 8 and their geometric means is 5 times their harmonic mean. Find the numbers.

12. The pth term of an H. P. is qr, and the qth term is pr. Find the rth term.

13. If the mth term of an H. P. is n and the nth term is m ; prove that the rth term is $\frac{mn}{r}$. (*P. U.* 1929)

14. The $(p-q)$th and $(p+q)$th terms of an G. P. are the A. M. and H. M. between a and b ; find the pth term of the G.P

15. If a, b, c are in G. P. show that $a+b, 2b, b+c$ are in H. P.

16 If a^2, b^2, c^2 are in A. P. then $b+c, b+a$, and $a+b$ are H P.

17. If a, b, c are in A. P. and b, c, d in H. P. then $ad=bc$.

18. If a, b, c are in H. P. then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in H. P.

19. If $\frac{p-x}{px}=\frac{q-y}{qy}=\frac{r-z}{rz}$ and p, q, r are in H. P., show that x, y, z are in H. P.

20. If a, b, c are the pth, qth, rth terms of an H. P. then $bc(q-r)+ca(r-p)+ab(p-q)=0$.

21. If a, b, c, be in A. P., p, q, r in H. P. and ap, bq, cr in G. P. then show that

$$\frac{p}{r}+\frac{r}{p}=\frac{a}{c}+\frac{c}{a} \qquad (P. U. 1943)$$

22. A G. P. and an H. P. have a, b, c, as their pth, qth and rth terms respectively, show that

$$a^{a(b-c)}b^{b(c-a)}c^{c(a-b)}=1.$$

Hence show that for the same conditions,

$a(b-c)\log a+b(c-a)\log b+c(a-b)\log c=0$. (P. U. 1939)

23. If a, b, c are in H. P., show that $(b+c-a)^2$, $(c+a-b)^2$ $(a+b-c)^2$, are in A. P.

24. Show that, if the equation

$$a(b-c)x^2+b(c-a)x+c(a-b)=0$$

has equal roots, then a, b, c are in H P.

25 If a, b, c be in G P ; a, p, b are in A P. b, q, c are in A. P. Prove that p, b, q are in H. P.

CHAPTER VII

Miscellaneous Series.

7·1. *To find the sum of the first n natural numbers.*

Let S denote the sum of the first n natural numbers, so that

$$S=1+2+3+\dots\dots+n.$$

Now the series on the right-hand side is an A. P., whose first term is 1, last term is n and the number of terms is also n. Hence summing up the A. P. on the right-hand side, we have

$$S=\frac{n}{2}(1+n)=\frac{n(n+1)}{2}.$$

7·2. *To find the sum of the squares of the first n natural numbers.*

$$S=1^2+2^2+3^2+\dots+n^2$$

We have identically

$$n^3-(n-1)^3=3n^2-3n+1$$

Changing n into $n-1$, $n-2$,..., 3, 2, 1, we get

$$\begin{array}{l}
(n-1)^3-(n-2)^3=3(n-1)^2-3(n-1)+1\\
(n-2)^3-(n-3)^3=3(n-2)^2-3(n-2)+1\\
\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\\
\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\\
\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots
\end{array}$$

$$\begin{array}{ccccccccc}
3^3 & - & 2^3 & = & 3\cdot3^2 & - & 3\cdot3 & + & 1\\
2^3 & - & 1^3 & = & 3\cdot2^2 & - & 3\cdot2 & + & 1\\
1^3 & - & 0^3 & = & 3\cdot1^2 & - & 3\cdot1 & + & 1
\end{array}$$

$\therefore$ By adding column-wise, we have

$$n^3=3S-3(1+2+\dots+n)+n$$

$$=3S-3.\frac{n(n+1)}{2}+n$$

$$\therefore\quad 3S=n^3+3.\frac{n(n+1)}{2}-n$$

$$=\frac{n(n+1)(2n+1)}{2}$$

Hence $S=\dfrac{n(n+1)(2n+1)}{6}$.

7·3. *To find the sum of the cubes of the first n natural numbers.*

Let S denote the required sum, so that

$$S=1^3+2^3+3^3+\ldots+n^3$$

As before, we have indentically

$$\begin{array}{llllll}
n^4 & -(n-1)^4=4n^3 & -6n^2 & +4n & -1 \\
(n-1)^4-(n-2)^4 & =4(n-1)^3 & -6(n-1)^2 & +4(n-1) & -1 \\
(n-2)^4-(n-3)^4 & =4(n-2)^3 & -6(n-2)^2 & +4(n-2) & -1 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
3^4-2^4 & =4\cdot3^3 & -6\cdot3^2 & +4\cdot3 & -1 \\
2^4-1^4 & =4\cdot2^3 & -6\cdot2^2 & +4\cdot2 & -1 \\
1^4-0^4 & =4\cdot1^3 & -6\cdot1^2 & +4\cdot1 & -1
\end{array}$$

$\because$ By adding column-wise, we have

$$n^4=4.S-6(1^2+2^2+\ldots+n^2)+4(1+2+\ldots+n)-n$$

$$=4.S-6\cdot\frac{n(n+1)(2n+1)}{6}+\frac{4n(n+1)}{2}-n$$

$$\because \quad S=\frac{n^4+n(n+1)(1n+1)-2n(n+1)+n}{4}$$

$$=\left\{\frac{n(n+1)}{2}\right\}^2.$$

7·4. Notation. It is convenient to denote the sums obtained in the last three articles by Σn, Σn^2 and Σn^3 respectively. Thus

Σn=sum of the first n natural number$=\dfrac{n(n+1)}{2}$

Σn^2=sum of the squares of the first n natural numbers

$$=\frac{n(n+1)(2n+1)}{6},$$

and Σn^3= sum of the cubes of first n natural numbers

$$=\left\{\frac{n(n+1)}{2}\right\}^2$$

It is important to note that $\Sigma n^3=(\Sigma n)^2$.

7·5. *To find the sum of n terms of a series whose nth term is given by* $T_n = an^3 + bn^2 + cn + d$.

We have $T_n = an^3 \quad + bn^2 \quad + cn \quad + d$

Changing n to $n-1, n-2, n-3, \ldots\ldots 3, 2, 1$ respectively

$$T_{n-1} = a(n-1)^3 + b(n-1)^2 + c(n-1) + d$$
$$T_{n-2} = a(n-2)^3 + b(n-2)^2 + c(n-2) + d$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$T_3 \quad = a.3^3 \quad + b.3^2 \quad + c.3 \quad + d$$
$$T_2 \quad = a.2^3 \quad + b.2^2 \quad + c.2 \quad + d$$
$$T_1 \quad = a.1^3 \quad + b.1^2 \quad + c.1 \quad + d$$

Hence by adding column-wise,

$$S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_{n-1} + T_n$$
$$= a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + dn.$$

7·6. Examples :—

1. Sum upto n terms the series

$$3(2^2+3^2)+4(3^2+4^2)+5(4^2+5^2)+\ldots\ldots$$

Here $T_n = (n+2)[(n+1)^2+(n+2)^2]$

$$= (n+2)(2n^2+6n+5)$$
$$= (2n^3+10n^2+17n+10)$$

$\therefore$ $S_n = 2\Sigma n^3 + 10\Sigma n^2 + 17\Sigma n + 10n$

$$= 2.\frac{n^2(n+1)^2}{4} + 10.\ \frac{n(n+1)(2n+1)}{6}$$
$$+ 17.\frac{n(n+1)}{2} + 10n$$
$$= \frac{n}{6}[3n(n+1)^2 + 10(n+1)(2n+1) + 51(n+1) + 60$$
$$= \frac{n}{6}[3n^3 + 26n^2 + 144n + 61].$$

2. Prove that the sum of the cubes of a number of consecutive integers is divisible by their sum.

Let the consecutive integers be

$$p+1, p+2, p+3, \ldots\ldots p+n$$

Their sum $= \frac{n}{2}(p+1+p+n) = \frac{n(2p+n+1)}{2}$ ......(*i*)

Sum of their cubes

$$=1^3+2^3+\ldots\ldots+(p+n)^3$$
$$-(1^3+2^3\ldots\ldots+p^3)$$
$$=\left[\frac{(p+n)(p+n+1)}{2}\right]^2-\left[\frac{p(p+1)}{2}\right]^2$$
$$=\left[\frac{(p+n)(p+n+1)}{2}+\frac{p(p+1)}{2}\right]$$
$$\times\left[\frac{(p+n)(p+n+1)}{2}-\frac{p(p+1)}{2}\right]$$
$$=\frac{2p(p+1)+2np+n(n+1)}{2}\times\frac{2np+n(n+1)}{2}$$
$$=\left[p(p+1)+np+\frac{n(n+1)}{2}\right]\times\frac{n(2p+n+1)}{2}\ \ldots(ii)$$

Now $\frac{n(n+1)}{2}$ is necessarily an integer ; therefore (i) and (ii) prove the proposition.

Exercises

Sum up to n terms the following series :—

1. $3.5+5.7+7.9+\ldots,\ldots,$ (*P. U.* 1919)
2. $1.4.7+2.5.8+3.6.9+\ldots\ldots$ (*P. U.* 1923)
3. $1.2^2+2.3^2+3.4^2\ldots\ldots$ (*P. U.* 1926)
4. $1^2+3^2+5^2+\ldots\ldots$ (*P. U.* 1929)
5. $2.5+5.8+8.11+\ldots\ldots$ (*P. U.* 1934)
6. $(1\frac{1}{3})^2+(2\frac{1}{3})^2+(3\frac{1}{3})^2+\ldots\ldots$ (*P. U.* 1935)
7. $1^2+4^2+7^2+\ldots\ldots$ (*Allahabad*, 1940)
8. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots\ldots$ (*P U.* 1942)
9. $1.n+2(n-1)+3(n-2)+\ldots+n.1$ (*P. U.* 1940)
10. Sum up to $3n$ terms
    $1+3-5+7+9-11+$ (*P. U.* 1941)

11. If $S_1, S_2\ldots\ldots, S_r$ denote the sum upto n terms of r Arithmetic series, whose first terms are $1, 2,\ldots, r$ respectively and whose common differences are $1, 3, 5,\ldots\ldots\ldots\ldots(2r-1)$ respectively, prove that $S_1+S_2+\ldots+S_r$ is the same as the sum of the first nr natural numbers.

12. The nth term of a series is $3n+3^n$; find its sum upto n terms.

13. The nth term of a series is $(-1)^n n^3$. Find its sum upto n terms.

14. Prove that the sum of first n odd integers is a perfect square.

15. Find the sum of the products of the first n natural numbers taken two at a time.

7·7. Method of Differences. In certain series the form of the nth term is not so apparent and before proceeding to find the sum of n terms we have to find the nth term. This may be done by the method illustrated by the following examples :—

Ex. 1. Find the nth term and the sum of first n terms of the series

$$1+5+11+19+29+\ldots\ldots$$

Let S denote the required sum so that

$$S=1+5+11+19+29+\ldots\ldots+T_{n-1}+T_n$$

Also $S= \quad 1+5+11+19+\ldots\ldots\ldots\ldots\ldots\ldots\ldots+T_{n-1}+T_n$

$\therefore$ By subtraction,

$$0=1+(4+6+8+10+\ldots\ldots\text{upto }\overline{n-1}\text{ terms})-T_n$$

$$\therefore\ T_n=1+(4+6+8+\ldots\ldots\text{upto }\overline{n-1}\text{ terms})$$

$$=1+\frac{n-1}{2}[2\times4+(n-2).2]$$

$$=1+(n-1)(n+2)$$

$$=n^2+n-1$$

$$\therefore\ S_n=\Sigma T_n=\Sigma n^2+\Sigma n-n$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}-n$$

$$=\frac{n}{6}(2n^2+6n-2)=\frac{n}{3}(n^2+3n-1).$$

Ex. 2. Find the nth term and sum upto n terms of the series

$$1+4+13+40+121+\ldots\ldots$$

Sol. Here the successive differences are in G. P.

Let $S=1+4+13+40+121+\ldots\ldots+T_{n-1}+T_n$

Also $S= \quad 1+4+13+40+\ldots\ldots \qquad +T_{n-1}+T_n$

$\therefore \quad 0=(1+3+9+27+81+\ldots\ldots \text{to } n \text{ terms})-T_n$

$\therefore \quad T_n=1+3+9+27+81+\ldots\ldots \text{to } n \text{ terms}$

$$=\frac{3^n-1}{3-1}=\tfrac{1}{2}.3^n-\tfrac{1}{2}$$

Changing n to $n-1$, $n-2, \ldots\ldots\ldots 3, 2, 1$ successively, we have $T_{n-1}=\frac{1}{2}\, 3^{n-1}-\frac{1}{2}$

$$T_{n-2}=\tfrac{1}{2}.3^{n-2}-\tfrac{1}{2}$$

....

....

$$T_3=\tfrac{1}{2}.3^3-\tfrac{1}{2}$$

$$T_2=\tfrac{1}{2}.3^2-\tfrac{1}{2}$$

$$T_1=\tfrac{1}{2}\,3-\tfrac{1}{2}.$$

$\therefore \; S=T_1+T_2+\ldots\ldots\ldots+T_n$

$$=\tfrac{1}{2}(3+3^2+\ldots\ldots\ldots+3^n)-\frac{n}{2}$$

$$=\tfrac{1}{2}.\, 3.\, \frac{3^n-1}{3-1}-\frac{n}{2}$$

$$=\tfrac{1}{4}\, 3^{n+1}-\frac{n}{2}-\frac{3}{4}.$$

Ex. 3. Sum to n terms the series

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots\ldots\ldots\ldots$$

We have $T_n=\dfrac{1}{(n\text{th term of } 1, 3, 5,\ldots)\times(n\text{th term of } 3, 5, 7..)}$

$$=\frac{1}{[1+(n-1).2][3+(n-1)2]}$$

$$=\frac{1}{(2n-1)(2n+1)}$$

So that, $T_n=\frac{1}{2}\left(\dfrac{1}{2n-1}-\dfrac{1}{2n+1}\right)$

$\therefore \quad T_{n-1}=\frac{1}{2}\left(\dfrac{1}{2n-3}-\dfrac{1}{2n-1}\right)$

$$T_{n-2}=\tfrac{1}{2}\left(\frac{1}{2n-5}-\frac{1}{2n-3}\right)$$

....

....

....

$T_3=\frac{1}{2}(\frac{1}{5}-\frac{1}{7})$
$T_2=\frac{1}{2}(\frac{1}{3}-\frac{1}{5})$
$T_1=\frac{1}{2}(1-\frac{1}{3})$

$\therefore\ S=T_1+T_2=\ldots\ldots+T_n$

$$=\frac{1}{2}\left[1-\frac{1}{2n+1}\right]=\frac{n}{2n+1}.$$

Ex. 4. Sum to infinity the series

$$\frac{1}{1.5}+\frac{1}{5.9}+\frac{1}{9.13}+\ldots\ldots\ldots$$

Here $T_n=\dfrac{1}{(4n-3)(4n+1)}$

$$=\frac{1}{4}\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)$$

Changing n into 1, 2, 3,........., we have

$T_1=\frac{1}{4}(1-\frac{1}{5})$
$T_2=\frac{1}{4}(\frac{1}{5}-\frac{1}{9})$
$T_3=\frac{1}{4}(\frac{1}{9}-\frac{1}{13})$

....
....
....

$\therefore$ Sum to infinity$=\frac{1}{4}(1-\frac{1}{5}+\frac{1}{5}-\frac{1}{9}+\frac{1}{9}-\frac{1}{13}+\ldots\ldots)$
$=\frac{1}{4}.$

Exercises.

Find the nth term and the sum up to n terms of the series :—

1. $1+3+6+10+\ldots\ldots\ldots$ (*P. U.* 1924)
2. $2+5+10+17+26+\ldots\ldots\ldots$ (*P. U.* 1920)
3. $4+11+22+37+56+\ldots\ldots\ldots$ (*Allahabad* 1926)
4. $3+5+9+17+\ldots\ldots\ldots$ (*Delhi* 940)
5. $1+5+13+29+61+\ldots\ldots\ldots$ (*P. U.* 1904)
6. $1+(2+3)+(4+5+6)+\ldots\ldots\ldots$
7. $1+(2+3)+(4+5+6+7)+(8+9+10+11+12+13+14+15)+\ldots\ldots\ldots$
8. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots\ldots\ldots$ (*P. U.* 1934)

9. $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\ldots\ldots\ldots$

10. $1+\dfrac{1}{1+2}+\dfrac{1}{1+2+3}+\ldots\ldots\ldots$ (*P. U.* 1938)

11. Find the sum to infinity in the above examples 8, 9, 10.

12. Natural numbers are written as

$$1,\ (2,\ 3),(4,\ 5,\ 6)\ldots\ldots\ldots$$

Show that the sum of the numbers in the nth group is $\frac{1}{2}n(n^2+1)$ (*P. U.* 1925)

13. Sum the series

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\ldots\ldots\ldots$$

to $2n$ terms.

14. Sum the series

$$\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+\ldots\ldots\ldots$$

to $3n$ terms.

15. Sum to n terms the series

$$\frac{1}{\sqrt{5}+1}+\frac{1}{\sqrt{9}+\sqrt{5}}+\frac{1}{\sqrt{13}+\sqrt{9}}+\ldots\ldots\ldots$$

to n terms. Deduce also the sum to infinity.

7·8. The Arithmetico-Geometric Series.

A series of the form

$$a+(a+d)r+(a+2d)r^2+\ldots\ldots+(a+\overline{n-1}d)r^{n-1}+\ldots\ldots$$

in which the first factors $a,\ a+d,\ a+2d,\ldots\ldots,\ a+\overline{n-1}d,\ldots\ldots\ldots$ from an A. P., and the second factors $1,\ r,\ r^2,\ldots\ldots\ldots,\ r^{n-1},\ldots\ldots\ldots$ form a G. P. is called an Arithmetico-Geometric Series. The formulae for its sum to n terms and for its sums to infinity need not be remembered. It is sufficient to remember the procedure only.

8·9. *To find the sum up to n terms of an Arithmetico-Geometric Progression.*

Let $S=a+(a+d)r+(a+2d)r^2+\ldots\ldots\ldots\ldots$

$$\ldots\ldots\ldots+(a+\overline{n-2}d)r^{n-2}+(a+\overline{n-1}d)r^{n-1}.$$

$\therefore \quad rS = ar + (a+d)r^2 + \ldots\ldots\ldots$
$\ldots\ldots\ldots + (a+\overline{n-2}d)r^{n-1} + (a+\overline{n-1}d)r^n.$

$\therefore$ by subtraction

$$(1-r)S = a + (dr + dr^2 + dr^3 + \ldots\ldots \text{to } \overline{n-1} \text{ terms}) - (a+\overline{n-1}d)r^n.$$

$$= a + dr\,\frac{1-r^{n-1}}{1-r} - (a+\overline{n-1}d)r^n.$$

$$\therefore \quad S = \frac{a}{1-r} + dr\frac{1-r^{n-1}}{(1-r)^2} - \frac{(a+\overline{n-1}d)r^n}{1-r}.$$

8·9. *To find the sum to infinity of an Arithmetico-Geometric Progression, when r is numerically < 1.*

Let S denote the sum to infinity, so that

$$S = a + (a+d)r + (a+2d)r^2 + \ldots\ldots \text{to } \infty.$$

$$\therefore \quad rS = \quad ar + (a+d)r^2 + \ldots\ldots \text{to } \infty.$$

$$\therefore (1-r)S = a + (dr + dr^2 + \ldots\ldots \text{to } \infty)$$

$$= a + \frac{dr}{1-r}$$

$$\therefore \quad S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}.$$

Ex. 1. Sum to n terms the series

$$1 + 5x + 9x^2 + 13x^3 + \ldots\ldots$$

Let $S = 1 + 5x + 9x^2 + 13x^3 + \ldots\ldots + (4n-7)x^{n-2} + (4n-3)x^{n-1}$

$\therefore \quad xS = x + 5x^2 + 9x^3 + \ldots\ldots + (4n-7)x^{n-1} + (4n+3)x^n$

$\therefore (1-x)S = 1 + (4x + 4x^2 + 4x^3 + \ldots\ldots \text{to } \overline{n-1} \text{ terms}) - (4n-3)x^n.$

$$= 1 + \frac{4x(1-x^{n-1})}{1-x} - (4n-3)x^n$$

$$\therefore \quad S = \frac{1}{1-x} + \frac{4x(1-x^{n-1})}{(1-x)^2} - \frac{(4n-3)x^n}{1-x}.$$

Ex. 2. Sum to n terms the series

$$1^2 + 2^2x + 3^2x^2 + 4^2x^3 + 5^2x^4 + \ldots\ldots$$

Here $S = 1^2 + 2^2x + 3^2x^2 - 4^2x^3 + 5^2x^4 + \ldots\ldots$
$\ldots + (n-1)^2x^{n-2} + n^2x^{n-1} \ldots (i)$

$\therefore \quad xS = 1^2.x + 2^2.x^2 + 3^2.x^3 + 4^2x^4 + \ldots\ldots$
$\ldots\ldots + (n-1)^2x^{n-1} + n^2.x^n \ldots (ii)$

Subtracting (*ii*) from (*i*) we get

$(1-x)S = 1^2 + (2^2-1^2)x + (3^2-2^2)x^2 + (4^2-3^2)x^3 + \ldots\ldots$
$\ldots\ldots + [n^2-(n-1)^2]x^{n-1} - n^2.x^n.$

$$=[1+3x+5x^2+7x^3+\ldots\ldots+(2n-1)x^{n-1}]-n^2.x^n. \quad (iii)$$

$$\therefore\ x(1-x)S=x+3x^2+4x^3+\ldots\ldots+(2n-1)x^n-n^2x^{n+1}\ldots\ldots\ldots (iv)$$

Subtracting (*iii*) from (*iv*) we obtain

$$\therefore\ (1-x)^2S=1+(2x+2x^2+3x^3+\ldots\ldots\text{to } \overline{n-1} \text{ terms})$$
$$-(n^2+2n-1)x^n+n^2.x^{n+1}$$

$$=1+\frac{2x(1-x^{n-1})}{1-x}-(n^2+2n-1)x^n+n^2.x^{n+1}$$

$$\therefore\quad S=\frac{1}{(1-x)^2}+\frac{2x(1-x^{n-1})}{(1-x)^3}-\frac{(n^2+2n-1)x^n}{(1-x)^2}+\frac{n^2.x^{n+1}}{(1-x)^2}.$$

Ex. 3. If S denotes the sum to infinity and S_n the sum to n terms of the series

$$2+2+\tfrac{3}{2}+1+\tfrac{5}{8}+\ldots\ldots;$$

find n for which $S-S_n \leqslant \frac{1}{2}$. (*P. U.* 1933)

Sol. The given series may be written as

$$1.(\tfrac{1}{2})^{-1}+2.(\tfrac{1}{2})^0+3(\tfrac{1}{2})+4(\tfrac{1}{2})^2+5(\tfrac{1}{2})^3+\ldots\ldots$$

Now $S_n=1.(\frac{1}{2})^{-1}+2(\frac{1}{2})^0+3.(\frac{1}{2})+4.(\frac{1}{2})^2+\ldots\ldots+(n-1)(\frac{1}{2})^{n-3}$
$+n.(\frac{1}{2})^{n-2}$

$\therefore\ \frac{1}{2}S_n=1(\frac{1}{2})^0+2(\frac{1}{2})+3.(\frac{1}{2})^2+\ldots\ldots+(n-1)(\frac{1}{2})^{n-2}+n.(\frac{1}{2})^{n-1}$

Hence by subtraction

$$\tfrac{1}{2}.S_n=[(\tfrac{1}{2})^{-1}+(\tfrac{1}{2})^0+(\tfrac{1}{2})+(\tfrac{1}{2})^2+\ldots\text{to } n \text{ terms})]-\frac{n}{2^{n-1}}$$

$$=(\tfrac{1}{2})^{-1}\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}-\frac{n}{2^{n-1}}$$

$$=4\left(1-\frac{1}{2^n}\right)-\frac{n}{2^{n-1}}=4-\frac{1}{2^{n-2}}-\frac{n}{2^{n-1}}$$

$$\therefore\quad S_n=8-\frac{1}{2^{n-3}}-\frac{n}{2^{n-2}}$$

Now as $n\to\infty$ $\frac{1}{2^{n-3}}$, $\frac{n}{2^{n-2}}$ both $\to 0$

$$\therefore\quad S=S_\infty=8$$

Hence $\quad S-S_n=\dfrac{1}{2^{n-3}}+\dfrac{n}{2^{n-2}}=\dfrac{n+2}{2^{n-2}}$

$\therefore\quad S-S_n \leqslant \frac{1}{2}$, if $\dfrac{n+2}{2^{n-2}} \leqslant \frac{1}{2}$

By putting $n=1, 2, 3,\ldots\ldots\ldots\ldots$in succession, we find that $S-S_n=\dfrac{n+2}{2^{n-2}}$ is $\leqslant \frac{1}{2}$, if $n \geqslant 6$.

Exercises

Sum up to n terms and to infinity (when $x<1$).

1. $1-x+2x^2-3x^3+4x^4 \ldots\ldots$ (*P. U.* 1932)
2. $1+4x+9x^2+16x^3+\ldots\ldots$
3. $1+3x+6x^2+10x^3+\ldots\ldots$
4. $1-\frac{3}{5}+\frac{5}{5^2}-\frac{7}{5^3}+\ldots\ldots$ (*Delhi*, 1932)
5. $1+\frac{2}{3}+\frac{3}{15}+\frac{4}{75}+\ldots\ldots$ (*P. U.* 1917)
6. $1^2+2^2x+3^2x^2+4^2x^3+\ldots\ldots$ (*P. U.* 1943)
7. $\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\frac{7}{16}+\ldots\ldots$ (*P. U.* 1942)
8. $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\ldots\ldots$ (*P. U.* 1926)
9. $1+(1+r)x+(1+r+r^2)x^2+\ldots\ldots$
10. $1-3(\frac{1}{4})+6(\frac{1}{4})^2-10(\frac{1}{4})^3+\ldots\ldots$ (*Allahabad* 1938)

Miscellaneous Exercise

Sum the following series to n terms :—

1. $1^3+3^3+5^3+7^3+\ldots\ldots$
2. $2.3^2+3.4^2+4.5^2+\ldots\ldots$
3. $2.3.5+3.4.6+4.5.7+\ldots\ldots$
4. $1+2.2+3.2^2+4.2^3+\ldots\ldots$
5. $1+(1+2)^2+(1+2+3)^2+\ldots\ldots$
6. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots\ldots$
7. $3+2+1+\frac{4}{9}+\frac{5}{27}+\frac{2}{27}+\ldots\ldots$
8. $(1+2)(1+\frac{1}{2})+(1+4)(1+\frac{1}{4})+(1+8)(1+\frac{1}{8})+\ldots\ldots$
9. $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)+(a^2+b^2)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+(a^3+b^3)\left(\frac{1}{a^3}+\frac{1}{b^3}\right)+\ldots$
10. $4+14+30+52+80+\ldots\ldots$
11. $1+11+31+61+101+\ldots\ldots$
12. $2+5+11+23+47+95+\ldots\ldots$
13. $6+19+58+175+526+\ldots\ldots$
14. $5+22+90+362+1450+\ldots\ldots$
15. $\frac{1}{a(a+2)}+\frac{1}{(a+2)(a+4)}+\frac{1}{(a+4)(a+6)}+\ldots\ldots$

Revision Questions II

1. Solve the quadratic equation $ax^2+bx+c=0$, and prove that it cannot have more than two distinct roots.

2. If α, β be the roots of the quadratic equation $ax^2+bx+c=0$ form an equation whose roots are $\alpha^2\beta$ and $\alpha\beta^2$.

3. If the equation $ax^2+bx+c=0$, remains unchanged when a, b, c are increased by the same quantity, prove that $x^2+x+1=0$

4. Discriminate the nature of the roots of a quadratic equation and find the value of m for which the expression $x^2+2x(1+3m)+7(3+2m)$ is a perfect square.

5. Solve the following equations :—

(*i*) $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}+\dfrac{x+c}{x-c}=3$

(*ii*) $\dfrac{1}{m+n+x}=\dfrac{1}{m}+\dfrac{1}{n}+\dfrac{1}{x}$

(*iii*) $x^4-x^3-4x^2+1=0$

(*iv*) $\dfrac{\sqrt{x+4}-\sqrt{x-4}}{\sqrt{x-4}+\sqrt{x+4}}=\dfrac{x}{4}$

(*v*) $\sqrt{x+y}+\sqrt{x-y}=\sqrt{2}$; $\sqrt{x^2+y^2}+\sqrt{x^2-y^2}=1$.

6. Show how to find the sum of a G. P., whose first term is a and common ratio is r. Deduce the sum to infinity when r is numerically less than 1.

7. Prove that the A. M., G. M., H. M., between any two positive quantities are in descending order of magnitude and themselves form a G. P.

8. Find x so that $a+x$, $b+x$, $c+x$ may be in H. P.

9. If a, b, c be in A. P., b, c, a in H. P., show that c, a, b are in G. P.

10. Sum the following series :—

(*i*) $2.5.8+5.8.11+8.11.14+$......to n terms.

(*ii*) $\left(1+\dfrac{1}{2}\right)+\left(2+\dfrac{1}{2^2}\right)+\left(3+\dfrac{1}{2^3}\right)+$...to n terms.

(*iii*) $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+$......to infinity.

(*iv*) $1+26+55+92+141$......to n terms.

(*v*) $6+66+666+$......to n terms.

(*vi*) $\cdot 2+\cdot 22+\cdot 222+$......to n terms.

(*vii*) $\dfrac{1}{x(x+2)}+\dfrac{1}{(x+2)(x+4)}+\dfrac{1}{(x+4)(x+6)}+$...to infinity.

Revision Exercises

I

1. Without actually solving it show that a quadratic equation cannot have more than two roots. (*P. U.* 1944).

2. Show that the roots of the equation $a^2x^2+b^2x+c^2=0$ are the squares of the roots of $ax^2+bx+c=0$, if a, b, c are in G. P.

3 Obtain the formula for the sum up to n terms of a G. P. and deduce the sum to infinity when the common ratio lies between -1 and $+1$.

4. If x_1, x_2 be the arithmetic means: y_1, y_2 the geometric means and z_1, z_2 the harmonic means between any two numbers: prove that $x_1+x_2 : z_1+z_2 : : y_1 y_2 : z_1 z_2$. (*P. U.* 1936)

5. Show that three quantities a b, c are in A. P., G. P. H. P. according as $\frac{a-b}{b-c}=\frac{a}{c}$ or $=\frac{a}{b}$ or $=\frac{a}{c}$.

6. Find the sum up to n terms of the following series and also deduce the sum upto infinity :—

(*i*) $\frac{1}{3 \cdot 5}+\frac{1}{5 \cdot 7}+\frac{1}{7 \cdot 9}+\ldots$

(*ii*) $1+2x+3x^2+4x^3+\ldots$ where $-1<x<+1$.

II

1. Discuss the nature of the roots of the quadratic equation $ax^2+bx+c=0$.

Also find the value of m, for which the equation

$$m^2x^2+2(m+1)x+4=0,$$

has equal roots

2. If α, β, be the roots of $ax^2+bx+c=0$, form an equation whose roots are $\frac{1}{(a\alpha+b)^2}$, $\frac{1}{(a\beta+b)^2}$.

3. Insert n arithmetic means between a and b and prove that their sum is n times the single arithmetic mean between a and b.

4. A, G, H represents the arithmetic, the geometric and the harmonic means between a and b, prove that

(*i*) A$>$G$>$H

and (*ii*) A, G, H are in G. P.

5. If $b+c$, $c+a$, $a+b$ are in H. P., show that a^2, b^3, c^2 are in A. P. (*P. U.* 1905)

6. Evaluate $1+4x+9x^2+16x^3+25x^4+\ldots$ to infinity for $x=\frac{1}{5}$. (*P U.* 1928)

7. Sum to n terms the series $1-3+6-10+25-21+\ldots$

III

1. Solve the equation (*i*) $\sqrt{x+2}+\sqrt{x+3}=5$

$$(ii)\ \frac{x+2}{x+1}+\frac{x-2}{x-1}=\frac{2x+1}{2x-2}.$$

2. Find the nth term of the series

$$4+4^2+4_1{}^3{}_3+\ldots \qquad (P.\ U.\ 1918)$$

3. If the mth term of an A. P. be n and the nth term be m, prove that the rth term $=\frac{mn}{r}$. (*P. U.* 1929)

4. If m is the Arithmetic mean of n consecutive integers show that the sum of their cubes is

$$mn[m^2+\tfrac{1}{4}(n^2-1)] \qquad (P.\ U.\ 1927)$$

5. Sum to infinity the series

$$1-x+2x^2-3x^3+4x^4-\ldots\ldots \qquad (P.\ U.\ 1929)$$

6. The sum of the three numbers is G. P. is 14 and the sum of their squares is 84, find the numbers.

7. If the sum to n terms of a series is an expression of the second degree in n; prove that the series is an A. P.

IV

1. Solve :— $\frac{x+1}{x+2}+\frac{x+2}{x+3}=\frac{2x+11}{x+6}$.

2. The natural numbers are written as

1
2, 3
4, 5, 6
7, 8, 9, 10
....

Show that the sum of the numbers in the nth row is $\frac{1}{2}n(n^2+1)$.

3. If a, b, c are in G. P., and $a+x$, $b+x$, $c+x$ in H. P., prove that $x=b$.

4. If a, b, c are in G. P. and x, y be the Arithmetic means between a, b, and b, c respectively, prove that

(i) $\frac{a}{x}+\frac{c}{y}=2$

(ii) x, b, y are in H. P.

5. A G. P. and an H. P. have the same pth, qth and rth terms a, b, c respectively. Prove that

$$a(b-c)\log a+b(c-a)\log b+c(a-b)\log c=0$$

(*P. U.* 1939)

6. Three numbers whose sum is 21 are in A. P., if 2, 3, 9 be added to them respectively, the results are in G. P. Find the numbers.

7. Sum the series

$$1.2.3+2.3.4+3.4\ 5+\ldots\ldots\text{to } n \text{ terms.}$$

V

1. Prove that (i) $(3+5w+3w^2)^6=64$, where w, w^2 denote the imaginary cube roots of unity.

(ii) $m^3-n^3=(m-n)(wm-w^2n)(w^2m-wn)$.

2. Solve the equations ;—

(i) $\frac{4}{x-6}-\frac{x-2}{x-3}=\frac{x+4}{x-5}-2.\frac{x-1}{x-4}$

(ii) $\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{5}{2}$; $x^2+y^2=90$. (*P. U.* 1902)

3. Euler says that it would be difficult to find x so as to make $p^4-2p^2x^2+x^4+x$ a perfect square, but show that it is not difficult. (*P. U.* 1926)

4. If a, b, c are in H. P. ; show that

$$\frac{1}{a}+\frac{1}{b+c}, \frac{1}{b}+\frac{1}{c+a}, \frac{1}{c}+\frac{1}{a+b}$$

are also in A. P.

5. If $y+z, z+x, x+y$ are in H. P., then

$$\frac{x}{y+z}, \frac{y}{z+x}, \frac{z}{x+y}$$

are in A. P (*Allahabad* 1931)

6. It $a^x=b^y=c^z$ and a, b, c are in G. P., show that x, y, z are in H P.

7. Sum the series

$$x+2\left(x-\frac{1}{x-1}\right)+3\left(x-\frac{2}{x-1}\right)+4\left(x-\frac{3}{x-1}\right)+\ldots\ldots$$

to n terms.

CHAPTER VIII

Permutations and Combinations

8·1. Def. 1. Each of the *sections* or *groups* that can be formed, by taking *some* or *all* of a number of given things is called a **combination**. The number of combinations of n *different* things taken r at a time is denoted by ${}^nC_r$.

Def. 2. Each of the *arrangements* that can be made by taking *some* or *all* of a number of things is called a **permutation.** The number of permutations of n different things taken r at a time is denoted by ${}^nP_r$.

The student must carefully note the basic distinction between a *combination* and a *permutation*. In a combination the *order* in which the things constituting the combination stand is immaterial. In other words an alteration in the *order* of the constituents of a combination does not change the combination on the other hand a permutation would change; if the order of its constituents were changed.

Before obtaining general formulae to evaluate ${}^nP_r$ and ${}^nC_r$, we solve a few simple examples from first principles. These must be carefully studied.

Ex. 1. The walls of a room contain 8 doors. In how many ways can a man enter the room ?

Sol. Any of the doors might be used in order to enter the room. Therefore there are as many ways of entering the room as the number of doors. Hence there are 8 ways of entering the room.

Ex. 2. There are 8 doors and 6 windows in the walls of a room. I wish to enter the room through a door and come out through a window. In how many ways can I do this ?

Sol. As in the above example, I can enter the room in 8 different ways. When I have entered the room in any one of these 8 ways, I can come out in 6 different ways, for I may use any one of the six windows for the purpose of coming out.

Thus there are 8 ways of entering the room and corresponding to each way of entering the room, there are 6 ways of coming out of it. Hence there are $8 \times 6 = 48$ ways of entering and coming out of the room.

Ex. 3. There are 8 doors leading to my room. In how many ways can I enter the room and come out by a different doors ?

Sol. As in Ex. 1, there are 8 different ways of entering the room. When this has been done in any one of the 8 ways, there are 7 doors left, through any of which I may come out.

Thus there are 8 ways of entering the room and corresponding to each way of entering the room, there are 7 ways of coming out. Hence the total number required $= 8 \times 7 = 56$.

Ex. 4. How many words can be formed out of the letters of the word LAHORE, taking all the latters at a time when no letter is to be repeated ?

Sol. There are 6 letters at our disposal with which 6 places are to be filled up.

Now the first place can be filled in 6 different ways, as any one of the 6 letters may be made to occupy it.

When the first place has been filled in any one of the 6 ways, there are five letters left any one of which may be made to occupy the second place. The second place can therefore be filled in 5 different ways.

Thus there are 6 different ways of filling up the first place and corresponding to each way of filling up the first place, there are 5 ways of filling up the second. Hence there are 6×5 ways of filling the first two places.

Similarly there are $6 \times 5 \times 4$ ways of filling the first three places.

Proceeding in this way, we find that the required number of words

$$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

Ex. 5. How many words of 6 letters can be formed out of the letters of the word LAHORE, when each letter may be repeated any number of times ?

Sol. Here as in the above example, the first place can be filled in 6 different ways. When this has been done in any one

of the 6 ways, the second place can again be filled in 6 different ways, as the particular letter used in filling the first place may be repeated.

Thus there are $6 \times 6 = 6^2$ different ways of filling up the first first two places.

Similarly there are 6^3 different ways of filling up the first three places.

Continuing in this way we find that the number of words required$=6^6$.

Ex. 6. How many words beginning with L can be formed out of the letters of the word LAHORE, by taking all the letters at a time, when no letter is to be repeated ?

Sol. Here the first place may be filled only in one way, as only the letter L can be made to occupy it and there are 5 different ways of filling the second place. Thus there are 1×5 different ways of filling the first two places.

Proceeding as in the above examples there are $1 \times 5 \times 4 \times 3 \times 2 \times 1 = 120$, different ways of filling all the five places. Hence the required number of words$=120$.

Ex. 7. How many words can be formed by taking all the letters of the word LAHORE, when A and H are required to come together ?

Sol. In this case we may look upon A H as a single letter Thus there are 5 different letters to arrange. This can be done in $5 \times 4 \times 3 \times 2 \times 1 = 120$ different ways.

But when these 5 letters have been arranged in any one of the 120 ways, we may arrange A and H between themselves in 2 different ways. For they may occur together as AH or as HA.

Hence the total number of words required$=2 \times 120 = 240$.

Ex. 8. In how many ways may the letters of the word LAHORE be arranged between themselves so that the vowels remain separated *i.e.* no two vowels come together ?

Sol. There are three vowels and three consonants. Adopting the procedure of the above examples, the consonants can between themselves be arranged in $3 \times 2 \times 1 = 6$ different ways.

Now if C denotes a consonant, the three vowels may be placed in any 3 of the 4 places marked $\times$, in the scheme given below. This by following the argument of the above

$$\times C \times C \times C \times$$

examples may be done in $4 \times 3 \times 2 = 24$, different ways.

Thus there are 6 different ways of arranging the consonants and corresponding to each way of arranging the consonants, there are 24 different ways of arranging the vowels. The total number of arrangements is therefore $6 \times 24 = 144$.

Ex. 9. In how many ways can the letters of the word LAHORE be arranged so that the consonants occupy the odd places and vowels occupy the even places ?

Sol. There are 3 consonants with which we have to fill three places *viz.*, the 1st, 3rd, and 5th. This can be done in $3 \times 2 \times 1 = 6$ different ways. The vowels can similarly be arranged in 6 different ways.

Since each way of arranging the consonants can be associated with each way of arranging the vowels, the total number of arrangements required is $6 \times 6 = 36$.

Ex. 10. How many *even* numbers greater than 5,000 can be formed by taking all the digits 2, 4, 5, 7 together ?

Sol. Since the number is to be even, only 2 or 4 can be made to occupy the unit's place. Thus there are 2 ways of filling up the unit's place.

Again, the number is to be greater than 5,000 ; therefore only 5 and 7 can be made to occupy thousands' place. Thus there are 2 different ways of filling the thousands' place.

The remaining two places *viz.*, the ten's and the thousand's have to be filled up by the remaining two digits. This can be done in 2 ways. Thus the required number of numbers $= 2 \times 2 \times 2 = 8$.

8·11. The **general principle** used in all the examples given above is ennunciated below :—

If one operation can be performed in m different ways and corresponding to each way of performing this operation, there are n different ways in which a second operation can be performed ; then the total number of ways in which the two operations can be performed in succession is $m \times n$.

The same principle may be ennunciated in a slightly different form :—

If an operation can be performed in m different ways and if each way of performing the first operation can be associated with each way of performing the second, then the total number of ways in which the two operations can be performed in succession is $m \times n$.

Exercises

1. There are 8 bus-services running between Lahore and Amritsar. In how many ways can a person go from Lahore to Amritsar and return by a different service.

2. Three prizes of different values are to be awarded to the speakers who stand first, second and third in a debating competition. If the number of competitors be 10, find the number of different ways in which the prizes might be given away.

3. How many different date-sheets are possible if the number of subjects to be examined is 5.

4. Write down actually all the permutations of the letters a, b, c, d and then answer the following questions :—

(*i*) What is the total number of permutation ?
(*ii*) How many permutations begin with a ?
(*iii*) How many of them end with d ?
(*iv*) How many of them begin with a and end with d ?
(*v*) In how many of them b and c come together ?

5. A gentleman has 8 collars, 12 neck-ties, 4 coats, 5 pants and 6 shirts. In how many different ways can he clothe himself ?

6. An officer has 3 inkpots, 4 pens, 5 pads and 2 writing tables. In how many ways can he begin to write a document ?

7. (*i*) What is the total number of words that can be formed by taking all the letters of the word PUNJAB at a time ?

(*ii*) How many of them begin with P ?
(*iii*) How many of them begin with P and end with B ?
(*iv*) In how many of them do the vowels come together ?
(*v*) In how many of them do the vowels remain separated ?

(*vi*) In how many of them do the vowels as well as the consonants remain together ?

(*vii*) How many of them begin with a consonant and end with a vowel ?

(*viii*) How many of them begin with a vowel and end with a vowel ?

(*ix*) How many of them begin with a consonant and end with a consonant.

(*x*) In how many of them N and J remain together ?

(*xi*) In how many of them N and J remain separated ?

8. (*i*) How many numbers can be formed by using all the digits 2, 4, 6, 5 ?

(*ii*) How many many of these numbers are even ?

(*iii*) How many of them are divisible by 2 ?

(*iv*) How many of them are divisible by 4 ?

(*v*) How many of them are divisible by 5 ?

(*vi*) How many of them are less than 4,000 ?

(*vii*) How many of them are greater than 4,000 but less than 6,000 ?

(*viii*) How many of them lie between 2400 and 2600 ?

(*ix*) How many of them lie between 2600 and 4500 ?

(*x*) How many of them are greater than 4500 ?

9. (*i*) How many numbers of 4 digits can be formed out of the six digits 0, 1, 2, 3, 4, 5 ?

(*ii*) How many of them will be divisible by 5 ?

(*iii*) How many of them will be divisible by 10 ?

(*iv*) How many of them will be divisible by 20 ?

(*v*) How many of them will be divisile by 25 ?

(*vi*) How many of them will be divisible by 50 ?

10. (*i*) In how many ways can six books on different subjects be arranged on a shelf ?

(*ii*) How many of these arrangements will begin with a particular book ?

(*iii*) How many of them will begin with a particular book and end with another particular book ?

(*iv*) In how many of them will two particular books always occur together ?

(*v*) In how many of them will two particular books always remain separated ?

11. In how many ways can a cricket eleven go out to bat ; if the two best players go out together to begin the game and the worst player is reserved for the end.

12. There are six seats in a car and one of them is reserved for the driver. In how many ways can the seats be occupied, if one of the persons refuses to sit in a particular seat ?

13. I have three books in German, 2 in English and 4 in French. In how many ways can I arrange these books on a shelf so that books of the same language remain together ?

14. Four persons enter a hall, where there are 12 vacant seats ? In how many ways can they seat themselves ?

15. There are 4 vacancies in a department and 12 applicants for them. In how many ways can the appointments be made.

16. Five gentlemen and three ladies apply in an office, where there are 3 vacancies. In how many ways can the appointments be made if one of the vacancies is to be filled by a lady and the other 2 by gentlemen ?

17. What is the possible number of appointments in the above question, if one of the posts is reserved for a lady, while there is no restriction on the other two ?

18. From 5 flags of different colours, how many signals can be given by hoisting them one above the other, when

(*i*) no flag is to be used more than once in a signal
(*ii*) when there is no such restriction.

19. In how many ways can 4 prizes be given away to 4 boys, when

(*i*) each boy is eligible for *all* the prizes ?

(*ii*) none of the boys is eligible for all the prizes ?

20. A question paper contain six questions, only three of which are to be attempted. In how many ways can a candidate attempt the paper, if one of his favourite questions is present, assuming that he attempts 3 questions.

8·2. Before proceeding to obtain a general formula for the value of ${}^nP_r$, it is convenient to introduce the reader to the following notation :

Def. Product of the first n natural numbers is denoted by $n\,!$ read as 'factorial n.'

Thus $1\,!=1$. $2\,!=1.2=2$, $3\,!=1.2.3=6$, $4\,!=1.2.3.4=24$. $5\,!=1.2.3.4.5=120$ and so on.

Ex. Prove the following :—

(*i*) $10.\,9\,!=10\,!$; (*ii*) $7.6\,5\,!=7\,!$

(*iii*) $10.9.8.=\dfrac{10\,!}{7\,!}$; (*iv*) $n(n-1).\,(n-2)\,!=n\,!$

(*v*) $n(n-1)(n-2)\ldots(n-r+1)=\dfrac{n\,!}{(n-r)\,!}$

(*vi*) $1.2.3\ldots50=1.3.5\ldots49.2^{25}.\,25\,!$

(*vii*) $1.2.3\ldots(2n-1).2n=1.3.5\ldots(2n-1).2^n.n\,!$

(*viii*) $1.2.3\ldots(4n-1)(4n).$

$$=1.3.5\ldots(4n-1).4^n.\,2n\,!$$
$$=1^2.3^2.5^2\ldots(2n-1)^2\times(2n+1)(2n+3)\ldots(4n-1)2^{3n}.$$

8·3. *To find the total number of possible arrangements of n dissimilar things taken r at a time i. e., to evaluate ${}^nP_r$.*

This is the same as the number of ways in which r places can be filled out of n dissimilar things.

Now the first place may be filled in n different ways, as any one of the n dissimilar things may be made to occupy it.

When the first place has been filled in any one of these n different ways, there are $n-1$ things left, any one of which may be made to occupy the second place. Therefore corresponding to each way of filling the first place there are $n-1$ ways of filling the second place. Therefore the total number of ways in which the first two places can be filled is $n(n-1)$.

When the first two places have been filled in any one of these ways, there are $n-2$ things left, so that the third place can be filled in $n-2$ ways. Therefore the total number of ways in which the first three places can be filled is $n(n-1)(n-2)$.

Similarly the total number of ways in which the first four places can be filled is $n(n-1)(n-2)(n-3)$.

Proceeding in this manner the total number of ways in which r places can be filled is

$$n(n-1)(n-1)(n-3)\ldots\text{upto } r \text{ factors}$$

$\therefore\ {}^nP_r = n(n-1)(n-2)\dots\dots\dots r$ factors
$= n(n-1)(n-2)\dots\dots\dots(n-r+1)$.

Cor. 1. $\displaystyle {}^nP_r = \frac{n(n-1)(n-2)\dots(n-r+1)\ (n-r)\,!}{(n-r)\,!} = \frac{(n)\,!}{(n-r)\,!}$

Cor. 2 In particular if $r=n$, we have

$$\begin{aligned} {}^nP_n &= n(n-1)(n-2)\dots(n-n+1) \\ &= n(n-1)(n-2)\dots3.2.1. \\ &= n\,! \end{aligned}$$

Cor. 3. Also from Cor. 1.

$${}^nP_n = \frac{(n)\,!}{(n-n)\,!} = \frac{n\,!}{0\,!}$$

$\therefore$ from Cor. 1 and Cor. 2.

$$n\,! = \frac{(n)\,!}{(0)\,!}$$

Hence we may take $0\,!$, which according to our definition of $n\,!$ has no meaning to be equivalent to 1.

Ex. 1. *Find the number of permutation of n dissimilar things taken r at a time which begin with a particular thing. Hence prove that ${}^nP_r = n \times {}^{n-1}P_{r-1}$. Deduce the formula for ${}^nP_r$ from this result.*

Sol. If all the permutations are to begin with a particular thing, this means that we are free only to fill the other $r-1$ places out of the remaining $n-1$ things which can obviously be done in ${}^{n-1}P_{r-1}$ ways.

Hence the number of permutations which begin with a particular thing is ${}^{n-1}P_{r-1}$.

Again, let the n things be denoted by $a, b, c\dots$

The number of those permutations which begin with a is ${}^{n-1}P_{r-1}$. The number of those which begin with b is also ${}^{n-1}P_{r-1}$, and similarly about each of the other things.

But obviously each permutation must begin with one or other of the n things. Hence the totat number of permutations is equal to $n \times {}^{n-1}P_{r-1}$.

$\therefore\ {}^nP_r = n \times {}^{n-1}P_{r-1}$.

Similarly ${}^{n-1}P_{r-1} = (n-1).\ {}^{n-2}P_{r-2}$.

$$^{n-2}P_{r-2}=(n-2)\,^{n-3}P_{r-3}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$^{n-r+3}P_3=(n-r+3).\,^{n-r+2}P_2$$
$$^{n-r+2}P_2=(n-r+2).\,^{n-r+1}P_1.$$

Multiplying the corresponding sides of all these equations and cancelling like factors on both sides, we have

$$^nP_r=n(n-1)(n-2)\ldots(n-r+2).\,^{n-r+1}P_1$$

Now $^{n-r+1}P_1$ is obviously equal to $n-r+1$.

$\therefore \quad {}^nP_r=n(n-1)(n-2)\ldots(n-r+2)(n-r+1).$

Ex. 2. *Find the number of permutations of n dissimilar things taken r at a time, when one of the n things is always to be (i) excluded (ii) included. Hence prove that*

$$^nP_r={}^{n-1}P_r+r\times{}^{n-1}P_{r-1}.$$

Sol. (*i*) If one of the things, say a, is to be excluded, this means that there are only $n-1$ things at our disposal and we are required to fill up r places out of them. This can be done in $^{n-1}P_r$ ways. Hence the number required in this case is $^{n-1}P_r$.

(*ii*) If a is always to be included, we have first of all to arrange the remaining $n-1$ things in $r-1$ places and then to insert a in each of the arrangements.

Now $n-r$ things can be arranged in $r-1$ places in $^{n-1}P_{r-1}$ ways. Also each of these permutations consists of $r-1$ things and there are r vacancies due to these $r-1$ things in any one of which a might be placed. Thus each of the $^{n-1}P_{r-1}$ permutations gives rise to r permutations containing a. Therefore the total number of permutations which contain a is $r\times{}^{n-1}P_{r-1}$.

Finally each permutation either includes a, or excludes it. Hence the total number of permutations of n things taking r at a time must be $^{n-1}P_r+r\times{}^{n-1}P_{r-1}$.

Thus $^nP_r={}^{n-1}P_r+r\times{}^{n-1}P_{r-1}$.

Ex. 3. *Find the number of permutations of n things taken r at a time when any of the things may be repeated any number of times.*

Sol. Here as in the ordinary case, the first place may be filled in n ways. But since any of the things may be repeated again the second place may also be filled in n ways.

Thus the total number of ways in which the first two places may be filled is $n \times n = n^2$.

Similarly the total number of ways in which the first three places may be filled is n^3 and so on.

Hence the total number of permutations of n things taken r at a time when the things may be repeated any number of times is n^r.

Ex. 4. *Prove from first principles that*

$$^nP_r = (n-r+1)\,{}^nP_{r-1}.$$

Hence find the value of ^nP_r.

Sol. We first of all arrange n things in $r-1$ places. This can be done in $^nP_{r-1}$ ways. There are $n-r+1$ things left with us, any of which may be made to occupy the rth place. Thus there are ^nP_r ways of filling the first $r-1$ places and corresponding to each way of doing this, there are $n-r+1$ ways of filling the rth places.

Hence the total number of ways of filling all the n places is $(n-r+1) \times {}^nP_{r-1}$.

$$\therefore \quad {}^nP_r = (n-r+1)\,{}^nP_{r-1}$$

Similarly $^nP_{r-1} = (n-r+2)\,{}^nP_{r-2}$

$$^nP_{r-2} = (n-r+3)\,{}^nP_{r-3}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$^nP_3 = (n-2).{}^nP_2$$

$$^nP_2 = (n-1).{}^nP_1$$

Also $\quad {}^nP_1 = n.$

$\therefore$ multiplying the corresponding sides of all these equations and cancelling like factors on both sides, we have

$$^nP_r = n(n-1)(n-2)\ldots(n-r+1).$$

Exercises

1. Find the value of n, when

 (*i*) $^nP_2 = 56$, (*ii*) $^nP_3 = 210$, (*iii*) $^nP_4 = 360$.

2. Find the value of n, when

 (*i*) $^nP_3 = 36 \times {}^{n-1}P_2$; (*ii*) $^nP_3 = 40 \times {}^{n-1}P_2$.

 (*iii*) $^{n+1}P_5 = 42 \times {}^{n-1}P_3$; (*iv*) $^nP_n = 5 \times {}^{n-1}P_{n-1}$.

3. Find the value of r, if

 (*i*) $^8P_r = 28 \times {}^{7}P_{11}$, (*ii*) $^{26}P_r = 36 \times {}^{35}P_{17}$.

 (*iii*) $^nP_r = (n-15) \times {}^nP_{r-1}$; (*iv*) $^{16}P_r = 10 \times {}^{15}P_{r-1}$.

4. Find the relation connecting n and r, if

(*i*) $2.{}^nP_r = r \times {}^nP_{r-1}$, (*ii*) ${}^{n+1}P_r = 2r.{}^nP_{r-1}$

(*iii*) ${}^nP_{2r} = (2n-r+1).\,{}^{n-1}P_{2r-1}$, (*iv*) $2.{}^nP_r = 3r \times {}^{n-1}P_{r-1}$

5. Find the value of n, when

(*i*) ${}^nP_r = 56 \times {}^{n-2}P_{r-2}$

(*ii*) ${}^nP_r = 120 \times {}^{n-3}P_{r-3}$

(*iii*) ${}^nP_r = {}^8P_r + r \times {}^8P_{r-1}$

6. Find the value of r, when

(*i*) ${}^{56}P_r = {}^{55}P_r + 2 \times {}^{55}P_{r-1}$

(*ii*) ${}^{40}P_{2r} = {}^{39}P_{2r} + 15 \times {}^{39}P_{2r-1}$.

7. Prove that the number of permutations of n things taken r at a time, if two particular things are always to be excluded is ${}^{n-2}P_r$.

8 Prove that the number of permutations of n things taken r at a time when one particular thing is always to be excluded and another particular thing is to be included is

$$r \times {}^{n-2}P_{r-1}.$$

9. Prove that the number of permutations of n things taken r at a time, when two particular things are always to be included is $r(r-1)\,{}^{n-2}P_{r-2}$.

10. Prove *ab initio*

$${}^nP_r = {}^{n-2}P_r + 2r.{}^{n-2}P_{r-1} + r(r-1)\,{}^{n-2}P_{r-2}.$$

11. Prove that the number of permutations of n things taken r at a time, when three particular things are always to be excluded is ${}^{n-3}P_r$.

12. The number of permutations of n things taken r at a time, when one of three particular things is always to be included, and the other two excluded is $3r \times {}^{n-3}P_{r-1}$.

13. The number of permutations of n things taken r at a time, when 2 of 3 particular things are always to be included and the other one excluded is $6r(r-1) \times {}^{n-3}P_{r-2}$.

14. The number of permutations of n things taken r at a time when 3 particular things are always to be included is

$$r(r-1)(r-2) \times {}^{n-3}P_{r-3}$$

15. Prove from first principles,

$$ {}^{n}P_r = {}^{n-3}P_r + 3r.{}^{n-3}P_{r-1} + 6r(r-1).{}^{n-3}P_{r-2} + r(r-1)(r-2){}^{n-3}P_{r-3}. $$

16. Prove from first principles,

$$ {}^{n}P_r = (n-r+2)(n-r+1).{}^{n}P_{r-2}. $$

17. In how many ways can 20 books be arranged on a shelf so that a particular pair of books shall not come together. (*P. U.* 1905)

18. Eight papers are set in an examination, two of them being of Mathematics. In how many ways can the papers be given, provided that the two Mathematics papers are not successive. (*P. U.* 1928)

19. How many different numbers of six digits can be formed with the digits 3, 1, 7, 0, 9, 5 ? How many of them will have zero in the ten's place. (*P. U.* 1943)

20. How many words can be formed with the letters of the word " article" so that the vowels occupy the even places. (*P. U.* 1942)

21. Find the number of ways in which m boys and n girls can be arranged in a row so that no two of the girls are together $(m>n)$ (*P. U.* 1935)

22. The figures 1, 2, 3, 4, 5 are written in every possible manner. How many of the numbers so formed will be greater than 23,000 ? (*Allahabad* 1939)

23. How many numbers can be formed by using the digits 0, 1, 2, 3, 4, each being used only once in each number. (*Dehli*, 1932)

24. In how many of the arrangements of 12 books taken 7 at a time will 4 specified books be (*i*) always included and (*ii*) always excluded. (*Dehli* 1931)

25. In how many of the permutations of n dissimilar things taken r at a time will four given things be excluded. (*P. U.* 1943)

8·4. *To find the number of permutation of n things taken all at a time, when p of the things are alike and of one kind, q of them are alike and of another kind and the rest all different.*

Let each of the p like things be denoted by a and each of the q like things by b and suppose that x is the required number of permutations.

In each of these x permutations replace the p like things a by $a_1, a_2, \ldots\ldots, a_p$, which are different from each other and different from all the remaining things. Now $a_1, a_2, \ldots\ldots, a_p$ can be permuted between themselves in $p\,!$ ways, so that if this change be made, each of the x permutations will give rise to $p\,!$ permutations.

Hence the total number of permutations of n things taken all at a time, when p of the things are alike and the rest all different would be $x \,.\, p\,!$.

Again in each of the $x \,.\, p\,!$ permutation, replace the q like things by $b_1, b_2, \ldots\ldots b_q$, which are different from each other and different from all the remaining things. Since these can be arranged between themselves in $q\,!$ ways therefore each of the $x \,.\, p\,!$ permutations gives rise to $q\,!$ permutions, when the above mentioned change is made.

Hence the total number of permutations of n things taking all at a time would now be $x.\, p\,!\, q\,!$.

But now all the things are dissimilar and therefore the number of permutations should be ${}^nP_n = n\,!$.

$$\therefore \quad x \,.\, p\,!\, q\,! = n\,!.$$

$$\therefore \quad x = \frac{n\,!}{p\,!\, q\,!}.$$

Ex. 1. How many words can be formed by rearranging the letters of the word 'Committee'.

Sol. The given word consists of 9 letters, but all of them are not dissimilar. Two *m's* are alike, two *t's* are alike, two *e's* are alike and the rest all different.

Hence the required number $= \dfrac{9\,!}{2\,!\,2\,!\,2\,!} = 45{,}360.$

Ex. 2. In how many ways can 6 one-rupee pieces and 5 one-anna pieces be arranged in a line.

Sol. Here there are 11 things to arrange. Out of these 6 are alike and of one kind and 5 are alike and of another kind.

Here the required number of ways$=\frac{11\,!}{6\,!\,5\,!}=462.$

Exercises.

1. (*i*) How many arrangements can be made out of the letters of the word ALLAHABAD ?

(*ii*) How many of these begin with A ?

(*iii*) How many of these begin with A and end with D ?

(*iv*) In how many of them do LL come together ?

(*v*) In how many of them do L and L remain separated ?

2. (*i*) How many arrangements can be made out of the letters of the INFINITE ?

(*ii*) How many of them begin with IN ?

(*iii*) How many of them begin with IN and end with TE ?

(*iv*) In how many of them do the three I's come together ?

(*v*) In how many of them do two I's come together while the third I remains separated from them ?

3. What is the total number of arrangements of the letters of the word :—

(*i*) ASSASSINATION, (*ii*) CALCUTTA,
(*iii*) BOMBARDMENT, (*iv*) OFFENSIVE,
(*v*) VESUVIUS, (*v*) SERIES.

4. (*i*) How many numbers of 5 digits can be formed by the digits 1, 1, 2, 2, 2 ?

(*ii*) How many of them will be less than 20,000 ?

(*iii*) How many of them will lie between 20,000 and 22,000 ?

5. (*i*) How many numbers of 4 digits can be formed by taking the digits 0, 2, 2, 4 ?

(*ii*) How many of them will be greater than 2,000 ?

(*iii*) How many of them will lie between 2,000 and 4,000 ?

6. How many numbers of 6 digits can be formed by taking the digits 0, 0, 1, 1, 2, 2 ?

7. How many numbers greater than 20,000,000 can be formed by the digits 1, 1, 1, 2, 3, 3, 4, 4 ?

COMBINATION

8·5. As has been pointed out before, the difference between a combination and a permutation is that in the former the order in which the constituents stand relative to each other is immaterial, while in the later it is very important. Thus *abc, acb, bca, bac, cab, cba* represent the same combination ; but different permutations.

To find the number of combinations of n dissimilar things taken r at a time, i. e., to evaluate ^nC_r.

Let x denote the required number of combinations. Now each of them consists of r things, which can be arranged between themselves in $r\,!$ ways. Thus each of the x combinations gives rise to $r\,!$ permutations. The total number of permutations, obtained by rearranging between themselves the constituents of each of the x combinations, is therefore $x.r\,!$.

But this process gives us all the ^nP_r permutations. Hence

$$x.\,r\,!={}^nP_r=\frac{n\,!}{(n-r!)}$$

$$\therefore \qquad x={}^nC_r=\frac{n\,!}{r\,!\,(n-r)!}.$$

Cor. 1. $^nC_r=\dfrac{n\,!}{r\,!\,(n-r)\,!}=\dfrac{n(n-1)\ldots(n-r+1)}{1.2\,3.\ldots\ldots\ldots\ldots r}.$

Cor. 2. $^nC_0=\dfrac{n\,!}{0\,!\,.\,n\,!}=\dfrac{1}{0\,!}=1.$

Cor. 3. *The product of any r consecutive integers is divisible by r !.*

Let n be the greatest of these integers.

Then from Cor. 1.

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r\,!}={}^nC_r.$$

Now ${}^nC_r$ is essentially an integer. Hence the result.

8.6. *To prove from first principles, that*

$${}^nC_r=\frac{n}{r}\times{}^{n-1}C_{r-1}$$

and hence to find the value of ${}^nC_r$ without using the formula for ${}^nP_r$.

Let the n things be represented by the letters $a, b, c, \ldots h$. Each of ${}^nC_r$ combinations contains r letters. Hence if all the combinations were to be written out completely the total number of letters thus written down would be

$$r\times{}^nC_r \qquad \ldots \quad (i)$$

Again consider a particular letter, say a. In order to obtain the combinations in which a always occurs: we have only to select $r-1$ things out of the remaining $n-1$, in all possible manners and then to insert a in each of these combinations. Now $r-1$ things can be selected out of $n-1$ things in ${}^{n-1}C_{r-1}$ ways. Hence the number of combinations in which a always occurs is ${}^{n-1}C_{r-1}$.

Thus when all the combinations are written down, a will have been written down ${}^{n-}C_{r-1}$ times.

Similarly b would be written down ${}^{n-1}C_{r-1}$ times and similar is the case about each of the n letters.

Hence the total number of letters used is also

$$\mathrm{n}\times{}^{n-1}C_{r-1} \qquad \ldots \quad (i)$$

Equating (i) and (ii)

$$r\times{}^nC_r=n\times{}^{n-1}C_{r-1}$$

i. e.

$${}^nC_r=\frac{n}{r}\times{}^{n-1}C_{r\ 1}$$

Similarly

$${}^{n-1}C_{r-1}=\frac{n-1}{r-1}\times{}^{n-2}C_{r-2}$$

$${}^{n-2}C_{r-2}=\frac{n-2}{r-2}\times{}^{n-3}C_{r-3}$$

$$^{n-r+3}C_3=\frac{n-r+3}{3}\times{}^{n-r+2}C_2$$

$$^{n-r+2}C_2=\frac{n-r+2}{2}\times{}^{n-r+1}C_1$$

Also $$^{n-r+1}C_1=\frac{n-r+1}{1}.$$

Multiplying the corresponding sides of all these equations and cancelling out common factors from both sides ; we get

$$^{n}C_r=\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots\ldots r}.$$

Ex. 1. In how many ways can a cricket eleven be chosen out of 13 players ?

Sol. The required number of ways is $^{13}C_{11}=78$.

Ex. 2. In how many ways can a cricket eleven be chosen out of 16 players, if two particular players are always to be (*i*) included, (*ii*) excluded.

Sol. (*i*) If two of the players are always to be included, we are really free to select only 9 out of the remaining 14 players. This can be done in $^{14}C_9$ ways, which is therefore the required number

(*ii*) If two of the players are always to be excluded, we have to select 11 players out of the remaining 14 and this can be done in $^{14}C_{11}$ ways which is therefore the required number.

Ex. 3. Out of 12 gentlemen and 6 ladies a committee is to be formed consisting of 4 gentlemen and 2 ladies. How many different committees are possible ?

Sol. Four gentlemen can be selected out of 12 in $^{12}C_4=\frac{12.11.10.9}{1.2.3.4}=495$ different ways.

Again 2 ladies can be selected out of 6 in $^{6}C_2=\frac{6.5}{1.2}=15$ different ways.

Also each way of selecting the gentlemen can be associated with each way of selecting the ladies. Therefore the required number $=495\times15=7425$.

Ex. 4. Find n and r, when

$$^{n}C_{r-1}:{}^{n}C_r:{}^{n}C_{r+1}=3:4:5.$$

Sol. We have ${}^nC_{r-1} : {}^nC_r = 3 : 4.$

$$\therefore \quad \frac{n\,!}{(r-1)\,!\,(n-r+1)\,!} : \frac{n\,!}{r\,!\,(n-r)\,!} = 3 : 4$$

$$\text{or} \quad \frac{n\,!}{(r-1)\,!\,.\,(n-r+1)\,(n-r)\,!} \times \frac{r\,(r-1)\,!\,.\,(n-r)\,!}{n\,!} = \tfrac{3}{4}$$

$$\text{or} \quad \frac{r}{n-r+1} = \frac{3}{4}$$

$$\therefore \quad 3n-7r+3=0 \qquad \ldots\ldots(i)$$

Also ${}^nC_r : {}^nC_{r+1} = 4 : 5$

$$\therefore \quad \frac{n\,!}{r\,!\,(n-r)\,!} \times \frac{(r+1)\,!\,(n-r-1)\,!}{n\,!} = \frac{4}{5}$$

$$\text{or} \quad \frac{r+1}{n-r} = \frac{4}{5}$$

$$\therefore \quad 3n-9r-5=0 \qquad \ldots\ldots(ii)$$

Solving (i) and (ii), we get

$$n=62, \; r=27.$$

Complementry Combinations

8 61. *To prove from first principles that*

$${}^nC_r = {}^nC_{n-r}.$$

Each time we select a group of r things, another group of $n-r$ things is invariably left behind. Thus to each way of selecting r things out of n, there always corresponds a way of selecting $n-r$ things. Hence the number of ways in which r things can be selected out of n is the same as the number of ways in which $n-r$ things can be selected out of the same number n

i. e., ${}^nC_r = {}^nC_{n-r}$

Otherwise thus :—

$${}^nC_{n-r} = \frac{n\,!}{(n-r)\,!\,[n-(n-r)]\,!} = \frac{n\,!}{(n-r)\,!\,.\,r\,!}$$

$$= \frac{n\,!}{r\,!\,(n-r)\,!} = {}^nC_r$$

Ex. 1. Evaluate ${}^{50}C_{48}$.

Sol. Here ${}^{50}C_{48}={}^{50}C_{50-48}={}^{50}C_2$

$$=\frac{50\times 49}{1.2}=1225$$

Ex. 2. Find n, if ${}^{n}C_{15}={}^{n}C_{14}$.

Sol. We know that if ${}^{n}C_{r}={}^{n}C_{r'}$,
then either $r=r'$ which is not the case here.

or $r'=n-r$.

$\therefore\quad 14=n-15 \quad$ or $\quad n=29$.

8·62. *To prove from first principles that*

$$^{n}C_{r}={}^{n-1}C_{r}+{}^{n-1}C_{r-1}.$$

All the combinations of n things taken r at a time can be divided into two mutually exclusive groups: firstly those that exclude a particular thing, say a, and secondly those that include a.

The number of those which exclude a is ${}^{n-1}C_{r}$; for in this case we are required to select r things out of the remaining $n-1$.

Similarly the number of those combinations which include a is ${}^{n-1}C_{r-1}$: for in this case we have to select only $r-1$ things out of the remaining $n-1$ things, one of these things *viz*, a having already been selected.

But the sum of the numbers of combinations in the two groups must be equal to the total number of combinations,

$\therefore\ {}^{n}C_{r}={}^{n-1}C_{r}+{}^{n-1}C_{r-1}$.

Otherwise thus :—

$$^{n-1}C_{r}+{}^{n-1}C_{r-1}=\frac{(n-1)\,!}{r\,!\,.\,(n-r-1)\,!}+\frac{(n-1)\,!}{(r-1)\,!\,(n-r)\,!}$$

$$=(n-1)\,!\left\{\frac{n-r}{r\,!\,(n-r)\,!}+\frac{r}{r\,!\,(n-r)\,!}\right\}$$

$$=(n-1)\,!.\frac{(n-r)+r}{r\,!\,(n-r)\,!}$$

$$=\frac{n.(n-1)\,!}{r\,!\,(n-r)!}=\frac{n\,!}{r\,!\,(n-r)\,!}$$

$$={}^{n}C_{r}.$$

Ex. 1. Find n, if ${}^nC_3+{}^nC_2=35$.

Sol. ${}^nC_2+{}^nC_3={}^{n+1}C_3=\dfrac{(n+1)n(n-1)}{1.2.3}$

$\therefore \quad \dfrac{(n+1)n(n-1)}{1.2.3}=35$

i. e., $(n+1)n(n-1)=35\times 6=7\times 6\times 5$

$\therefore \quad n=6.$

Ex. 2. Prove that

$${}^nC_0+{}^nC_1+{}^nC_2+\dots\dots+{}^nC_n=2^n$$

Sol. ${}^nC_0+{}^nC_1+{}^nC_2+\dots\dots+{}^nC_n$

$={}^{n-1}C_0+({}^{n-1}C_0+{}^{n-1}C_1)+({}^{n-1}C_1+{}^{n-1}C_2)+\dots+{}^{n-1}C_{n-1}$

($\because$ ${}^nC_0=1={}^{n-1}C_0$, ${}^nC_n=1={}^{n-1}C_{n-1}$
and ${}^nC_r={}^{n-1}C_{r-1}+{}^{n-1}C_r$)

$=2({}^{n-1}C_0+{}^{n-1}C_1+{}^{n-1}C_2+\dots+{}^{n-1}C_{n-1})$

$=2^2({}^{n-2}C_0+{}^{n-2}C_1+{}^{n-2}C_2+\dots+{}^{n-2}C_{n-2})$

$=2^3({}^{n-3}C_0+{}^{n-3}C_1+{}^{n-3}C_2+\dots+{}^{n-3}C_{n-3})$

$=\dots\dots\dots\dots\dots\dots$

$=\dots\dots\dots\dots\dots\dots$

$+2^{n-3}({}^3C_0+{}^3C_1+{}^3C_2+{}^3C_3)$

$=2^{n-2}({}^2C_0+{}^2C_1+{}^2C_2)$

$=2^{n-1}({}^1C_0+{}^1C_1)$

$=2^{n-1}(1+1)=2^n.$

8·63. To find the number of ways in which one, two, three or any number of things can be selected out of $p+q+r$ things, p of which are alike and of one kind, q are alike and of a second kind and r are alike and of a third kind.

The p alike things may be dealt with in $p+1$ ways, as we may select 0, 1, 2,..., p things out of them.

Similarly there are $q+1$ ways of dealing with the q alike things.

Since each way of dealing with the p things may be associated with each way of dealing with the q things, the number of ways in which the $p+q$ things can be dealt with is $(p+1)(q+1)$.

Similarly the number of ways in which all the $p+q+r$ things can be dealt with is $(p+1(q+1)(r+1)$.

But this includes the case in which none of the $p+q+r$ things is selected. Hence rejecting the single case the required number is $(p+1)(q+1)(r+1)-1$.

Cor. The number of ways in which a selection can be made out of n different things is $(1+1)(1+1)(1+1)$.........to n factors -1.

$=2^n-1$.

This is another proof of the result

$$^nC_0+{}^nC_1+{}^nC_2+\ldots\ldots+{}^nC_n=2^n.$$

Ex. 1. There are 4 apples and 3 oranges. Find the number of ways in which (*i*) at least one fruit, (*ii*) at least one fruit of each kind can be selected out of them.

Sol. (*i*) The 4 apples can be dealth with in 5 different ways, as we may select 0, 1, 2, 3 or 4 out of them. Similarly the 3 oranges may be dealt with in 4 ways.

Hence the total number of ways in which a selection can be made is $5\times4=20$.

But this includes the case in which neither an apple nor an orange is selected. Rejecting this case the required number $=20-1=19$.

(*ii*) In this case the apples may be dealt with in only 4 ways as we may select 1, 2, 3 or all of them. Similarly the oranges can be dealt with in 3 different ways.

Hence the required number$=4\times3=12$.

§ 64. *To find the number of ways in which $m+n$ things can be divided into groups of m and n.*

Each time we select a group of m things out of $m+n$, a group of n things is automatically formed. Hence we are only to find the number of ways in which m things can be selected out of $m+n$ things.

This can be done in ${}^{m+n}C_m=\dfrac{(m+n)\,!}{m\,!\,n\,!}$, ways.

Cor. The number of ways in which $m+n+p$ things can be divided into 3 groups of m, n and p things is similarly

$$\frac{(m+n+p)\,!}{m\,!\,.\,n\,!\,.\,p\,!}.$$

Note. Putting $m=n$, we see that the number of ways in which $2n$ things can be divided into two groups of n things each is $\dfrac{2n\,!}{(n\,!)^2}$. This will be so only, if the two groups into

which the things are to be divided are different. But if there is no distinction between the two *groups* as such, then the number will be only $\frac{2n\,!}{2\,!\,(n\,!)^2}$.

For in this case, it does not matter, whether a particular thing occurs in one group or the other, so long as its companions in the groups are the same. Therefore, we can fix up one of the $2n$ things in one of the groups and select $n-1$ companions for it out of the remaining $2n-1$ in all possible ways. This can be done in

$$^{2n-1}C_{n-1}=\frac{(2n-1)\,!}{(n-1)\,!\,n\,!}=\frac{2n\,!}{2\,!\,(n\,!)^2}\text{ ways.}$$

Similarly the number of ways in which $3n$ things can be divided into 3 similar groups of n things each

$$=\frac{3n\,!}{3\,!\,(n\,!)^3}.$$

Ex. 1. Find the number of ways in which 22 hockey players can be divided into teams of eleven members each to play against each other.

Sol. There is no essential distinction between the two teams, therefore the required number $=\frac{22\,!}{2\,!\,(11\,!)^2}$.

Ex 2. Find the number of ways in which 8 explorers can divide themselves into two equal parties, one of which is to go towards the North Pole and the other towards the South Pole.

Sol. In this case the two groups are different. Hence the required number $=\frac{8\,!}{(4\,!)^2}=\frac{8.7.6.5}{1.2.3.4}=70.$

8 7. *To find the values of r for which ^nC_r is greatest.*

It is easy to see that

$$^nC_r=\frac{n-r+1}{r}.{}^nC_{r-1}$$

$\therefore\ {}^nC_r$ is $> = \text{or} < {}^nC_{r-1}$ according as $\dfrac{n-r+1}{r} > = \text{or} < 1$

i.e., according as $r < = \text{or} > \dfrac{n+1}{2}$.

Case I. Let n be even $=2m$, so that ${}^nC_r > = \text{or} < {}^nC_{r-1}$, according as $r < = \text{or} > m+\frac{1}{2}$.

Hence putting $r=0, 1, 2, 3, \ldots, m$, we see that ${}^nC_0$, ${}^nC_1$, ${}^nC_2, \ldots, {}^nC_m$ are in ascending order of magnitude.

And putting $r=m+1$, $m+2, \ldots$, we see that ${}^nC_m$, ${}^nC_{m+1}$, ${}^nC_{m+2}, \ldots$ are in descending order of magnitude. Thus ${}^nC_r$ is greatest when $r=m=\dfrac{n}{2}$.

Case II Let n be odd $=2m+1$, so that ${}^nC_r > = \text{or} < {}^nC_{r-1}$ according as $r > = \text{or} < m+1$.

Hence putting $r=0, 1, 2, \ldots\ldots, m$, we see that ${}^nC_0$, ${}^nC_1$, ${}^nC_2, \ldots\ldots, {}^nC_m$ are in ascending order of magnitude.

Putting $r=m+1$, we see that ${}^nC_m = {}^nC_{m+1}$.

Finally putting $r=m+2$, $m+3 \ldots$ we see that ${}^nC_{m+1}$, ${}^nC_{m+2} \ldots$ are in descending order of magnitude. Thus in this case ${}^nC_r$ is max. when r is either m or $m+1$, *i. e.*, when r is either $\dfrac{n-1}{2}$ or $\dfrac{n+1}{2}$.

Miscellaneous Solved Examples

Ex. In how many different ways can 5 persons seat themselves around a table.

Sol. Let a, b, c, d, e denote the five persons and consider the arrangements *abcde, bcdea, cdeab, deabc, eabcd.* If the 5 persons were to seat themselves in a line, these would be 5 different arrangements But since the persons are required to seat themselves in a circle, these actually represent the same arrangement.

Thus a single *circular arrangement* gives rise to 5 *linear arrangements* Hence if x is the number of circular permutations required, we have $5x=5\,!$, *i. e.*, $x=4\,!$

Otherwise thus :—

In a circular arrangement, it is not the actual position of a person that matters. What matters is the position of a person *relative* to others. Hence we may fix up the position of one of the persons and then arrange the remaining 4 persons in all possible ways. Hence the required number is $=4\,!$

Note. (*i*) It can similarly be proved that the number of circular permutations of n different things taken all a time is $(n-1)\,!$

(*ii*) If we do not make any distinction between a clockwise and counter-clockwise arrangement of the same letters in the same sequence, the number of circular permutations will be $\frac{1}{2}(n-1)\,!$ and not $(n-1)\,!$ This can be easily proved.

Ex. In how many different ways can five persons seat themselves around a table so that there are no two arrangements in each of which each of the persons has the same neighbours.

Sol. Here we are not to make any distinction between clockwise and counter-clockwise arrangements arising out of the same persons taken in the same sequence.

Hence the number required$=\frac{1}{2}(5-1)\,!=12.$

Ex. How many different garlands can be made out of 7 flowers of different colours, taking all the flowers at a time.

Sol. Here again there is to be no distinction between the clockwise and counter-clockwise arrangements, as each can be obtained from the other by turning the garland upside down.

Hence the required number$=\frac{1}{2}(7-1)\,!=360.$

Ex In how many ways can 7 red and 5 white balls be placed in a row so that no two white balls are together. (Assuming that the red balls are all alike and the white balls are all alike.)

Sol Let the red balls be placed in a row. Then there are 6 places formed between the 7 red balls and 2 places at the ends, in each of which a white bail may be placed. Thus we have to select 5 places out of these 8.

This can be done in ${}^8C_5={}^8C_3=\frac{8.7\,6}{1.2.3}=56$ ways, which is therefore the required number.

Exercises.

1. From 7 Englishmen and 4 Indians a committee of 6 is to be formed, in how many ways may it be done,
   (*i*) when the committee contains exactly 2 Indians,
   (*ii*) at least 2 Indians.

2. I have 20 friends, 12 of whom are men and the remaining are ladies. In how many ways may I invite 15 guests from among them so that 8 out of them are men.

3. In how many ways can 6 red balls and 4 white balls be placed so that no 2 white balls are together ?

4. A man has 12 children. How many times can he take them to the zoo 4 at a time, so as never to take the same four twice.

5. From 17 consonants and 5 vowels how many words can be formed each consisting of 2 vowels and 3 consonants.

6. In how many ways can 12 students be divided into batches of 4 to play tennis ?

7. How many triangles can be formed by joining the angular points of a polygon of 18 sides ?

8. There are 50 subscribers to a telephone exchange. In how many ways can the subscribers be put into communications one with the other.

9. A father with 8 children takes them three at a time to the Zoological Gardens, so often as he can without taking the same three children together more than once. How often will he go ? How often will each child go ? (*Calcutta*)

10. Eight papers are to be set in an examination. In how many different orders can the papers be given, provided that the two mathematics papers are not succeeding. (*P. U.* 1914)

11. There are n points in a plane, no three of which are in the same straight line, with the exception of p which are all in the same straight line ; find the number of (*i*) straight lines, (*ii*) triangles that can be obtained by joining them.

12. If the number of combinations of n different things taken r at a time in which a particular thing does not occur be p times the number of those in which it does occur, prove that $n=(p+1)r$.

13. Find the total number of diagonals of a polygon of n sides.

14. In how many ways can 8 persons sit at a round table so that two particular persons may be (*i*) near each other, (*ii*) separate ?

15. Find the number of ways in which 7 ladies and 7 gentlemen can be placed alternately in a ring.

16. A person has 15 acquaintances of whom 5 are relatives. In how many ways can he invite 13 guests from among them so that 3 of them may be relatives ?

17. A gentleman invites a party of $m+n$ friends to dinner, and places m at one round table and n at another. Find the total number of ways in which he can arrange them.

18. If there are, in a bag, 10 white and 6 red balls, in how many different ways may 5 balls be drawn out, so that in each draw there may be at least 2 red balls.

19 A police post consisting of 5 mounted men and 9 foot policemen has to furnish a daily guard consisting of 2 from each class. How many days will elapse before the same guard recurs after all possible selections have been made.

20. In how many ways can 10 persons go in two boats, so that there may be five in each boat, supposing that two particular persons will not go in the same boat.

21. In how many ways can 5 rupees and 6 pice be arranged in a ring (the rupees being all alike and the pice all alike) ?

(*P. U* 1937)

22. If the number of combinations of n things taken r together be equal to the number of combinations of them $2r$ together ; and if the number of their combinations $(r+2)$ together be equal to $\frac{11}{3}$ times the number of their combinations $(r-1)$ together ; find n and r.

23. There are m men and n monkeys, $(n>m)$.

(*i*) Find the number of ways in which each man may become the master of a monkey.

(*ii*) If a man may have any number of monkeys, in how many ways may every monkey have a master ?

24. A cricket team of 11 players is to be selected from two groups consisting of six and eight players, respectively. In how many ways can the selection be made on the supposition that the first group shall contribute not fewer than four players.

25. An office requires 3 senior and 6 junior clerks. How many different selections can be made out of 5 applicants for the higher and 10 for the lower appointments ? How often might a particular junior find himself in the same arrangement with a particular senior ?

26. How many numbers of four digits can be formed in each of which every digit is greater than the one which follows it on the right ?

27. How many different sums can be formed with 7 ten-rupee notes, 12 five-rupee notes, 4 rupee coins and 13 eight-anna pieces.

28. In how many ways can 22 cricket players be divided into two elevens to play a match against each other.

29. In how many ways can 22 cricket players be divided into two elevens, one of which is to play against a team from Amritsar and another against a team from Jullundhar.

30. Show that the ratio of the number of combinations of $4n$ things taken $2n$ together to that of $2n$ taken n together is

$$\frac{1.3.5\ldots(4n-1)}{[1.3.5\ldots(2n-1)^2]}.$$

31. If ${}^{n+1}C_{r-1} : {}^{n}C_{r} : {}^{n-1}C_{r-1} = 11 : 6 : 3$, find n and r.

(*P. U.* 1936)

32. Given seven straight lines of lengths 1, 2, 3, 4, 5, 6, 7 inches respectively. Find the number of ways in which four can be selected to form a quadrilateral. (*P. U.* 1926)

33. In how many ways can 16 departments be allotted to four ministers so that no minister shall have less than three departments under him. (*P. U.* 1939)

34. Find the number of arrangements which can be made from the letters of the word '*algebra*', without altering the relative positions of the vowels or the consonants. (*P. U.* 1942)

35. There are 4 letters and 4 directed envelopes. In how many ways can the letters be put into envelopes so that each is put in a wrong envelope (*Allahabad* 1931)

36. How many words can be formed with the letters of the word AMRITSAR, taken all together ? (*P. U.* 1934)

37. Find how many arrangements can be made with the letters of the word "*mathematics*". In how many of these, the vowels come together ? (*P. U.* 1941)

38. The Premiers of eleven provinces of India meet to discuss the problem of minorities. In how many ways can they seat themselves at a round-table if the Punjab and Bengal Premiers choose to sit together ? (*P. U.* 1940)

39. Find the number of ways in which n different beads can be arranged to form a necklace. (*P. U.* 1942)

40. A gentleman invites a party of 13 guests to a dinner and places eight of them at one table and the remaining 5 at another, the tables being round. Find the number of ways in which he can arrange the guests. (*P. U.* 1942)

41. Out of 11 players, how many should be taken to form a group so that the number of different groups may be the greatest. (*Dehli*, 1933)

42. Prove that the greatest value of ${}^{2n}C_r$ is double of the greatest value of ${}^{2n-1}C_r$. (*P. U.* 1915)

43. On a New-year's day, every member of a family sends a card to every other and the postman delivers 156 cards, how many members are there in the family. (*P. U.* 1934)

44. A tennis tournament is to be played by 10 pairs of students. Each pair is to play every other pair one set. If four sets are played each day, how many days should be allowed for the tournament. (*P. U.* 1911)

45 From 7 Unionists and 4 Congressites a committee of 6 is to be chosen. In how many ways can this be done when the committee contains at least 2 Congressites. (*P.U.* 1938)

CHAPTER IX

BINOMIAL THEOREM

Positive Integral Index

9·1. Any expression consisting of two terms connected by the sign + or − is called a binomial expression or simply a binomial. The **Binomial Theorem** for a positive integral index states that

If n is a positive integer, then

$$(a+x)^n={}^nC_0.a^n+{}^nC_1.a^{n-1}x+{}^nC_2\,a^{n-2}x^2+{}^nC_3.a^{n-3}x^3+\ldots\ldots$$
$$\ldots+{}^nC_r.a^{n-r}x^r+\ldots+{}^nC_{n-1}\,ax^{n-1}+{}^nC_n.x^n.$$

Proof. For $n=1, 2, 3,$ we find by actual multiplication that

$$(a+x)^1=a+x={}^1C_0\,a+{}^1C_1x$$
$$(a+x)^2=a^2+2ax+x^2={}^2C_0a^2+{}^2C_1ax+{}^2C_2x^2$$
$$(a+x)^3=a^3+3a^2x+3ax^2+a^3$$
$$={}^3C_0a^3+{}^3C_1a^2x+{}^3C_2ax^2+{}^3C_3x^3.$$

Thus we see that the theorem is true for $n=1, 2, 3$.

Now let us suppose that the theorem is true for a particular value of n, say m, *i e.*,

$$(a+x)^m={}^mC_0a^m+{}^mC_1a^{m-1}x+{}^mC_2a^{m-2}x^2+\ldots$$
$${}^mC_ra^{m-r}x^r+\ldots+{}^mC_{m-1}ax^{m-1}+{}^mC_mx^m. \quad (1)$$

Multiply both sides of this equation first by a and then by x. We have

$$a(a+x)^m={}^mC_0a^{m+1}+{}^mC_1a^mx+{}^mC_2a^{m-1}x^2+\ldots$$
$$\ldots+{}^mC_ra^{m-r+1}x^r+\ldots+{}^mC_{m-1}a^2x^{m-1}+{}^mC_max^m$$

and
$$x(a+x)^m={}^mC_0a^mx+{}^mC_1a^{m-1}x^2+{}^mC_2a^{m-2}x^3+\ldots\ldots$$
$$\ldots+{}^mC_ra^{m-r}x^{r+1}+\ldots+{}^mC_{m-1}ax^m+{}^mC_mx^{m+1}$$

Therefore by addition

$$(a+x)^{m+1}={}^mC_0a^{m+1}+({}^mC_1+{}^mC_0)a^mx+({}^mC_2+{}^mC_1)a^{m-1}x^2$$
$$+({}^mC_3+{}^mC_2)a^{m-2}x^3+\ldots+({}^mC_r+{}^mC_{r-1})a^{m-r+1}x^r$$
$$+\ldots+({}^mC_m+{}^mC_{m-1})ax^m+{}^mC_mx^{m+1}$$

But ${}^mC_0=1={}^{m+1}C_0$ and ${}^mC_m=1={}^{m+1}C_{m+1}$.

Also ${}^mC_0+{}^mC_1={}^{m+1}C_1$, ${}^mC_2+{}^mC_1={}^{m+1}C_2$.

${}^mC_3+{}^mC_2={}^{m+1}C_3,\ldots\ldots$, and generally ${}^mC_r+{}^mC_{r-1}={}^{m+1}C_r$

$\therefore$ we have

$$(a+x)^{m+1} = {}^{m+1}C_0\, a^{m+1} + {}^{m+1}C_1 a^m x + {}^{m+1}C_2 a^{m-1} x^2$$
$$+ {}^{m+1}C_3 a^{m-2} x^3 + \ldots + {}^{m+1}C_r a^{m-r+1} x^r$$
$$+ \ldots + {}^{m+1}C_m a x^m + {}^{m+1}C_{m+1} x^{m+1}. \qquad (2)$$

Comparing (1) and (2), we see that (2) is of the same form as (1) ; only m has been changed to $m+1$.

Therefore, if the theorem is true for $n=m$, it is also true for $n=m+1$.

But we know already that the theorem is actually true for $n=1, 2, 3$; therefore the theorem is true for $n=3+1=4$. Again, since the theorem is true for $n=4$, it must be true for $n=4+1=5$ and so on. Thus the theorem is generally true.

Cor. Changing x to $-x$, we have

$$(a-x)^n = {}^nC_0 a^n - {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 - \ldots$$
$$\ldots + (-1)^r\, {}^nC_r a^{n-r} x^r \ldots + (-1)^n\, {}^nC_n x^n$$

so that the terms are alternately positive and negative, the last term being positive or negative, according as n is even or odd.

Cor. 2. Putting $a=1$, we have

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n$$

and $(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - \ldots + (-1)^r\, {}^nC_r x^r \ldots$

$$\ldots + (-1)^n\, {}^nC_n . x^n.$$

8·2. Properties of the Binomial Expansion.

(*i*) The number of terms in the binomial expansion of $(a+x)^n$ is $n+1$, *i.e.*, one more than the power to which $a+x$ is raised.

(*ii*) The power of a goes on decreasing by unity while that of x goes on increasing by unity, the sum of the two powers thus remaining constantly equal to n.

(*iii*) The successive co-efficients are ${}^nC_0$, ${}^nC_1$, ${}^nC_2, \ldots\ldots {}^nC_n$ and the power of x in any term is equal to the suffix of C in that term.

8·21. The general term.

If $T_1, T_2, T_3, \ldots\ldots$ denote the successive terms in the binomial expansion of $(a+x)^n$, we have

$$T_1 = {}^nC_0 a^{n-0} . x^0$$
$$T_2 = {}^nC_1 . a^{n-1} . x$$
$$T_3 = {}^nC_2 . a^{n-2} . x^2 \text{ and so on,}$$

generally $T_{r+1}={}^nC_r a^{n-r}x^r$.

${}^nC_r a^{n-r}x^r$ is called the general term in the expansion of $(a+x)^n$, because all other terms can be obtained from it by putting $r=0, 1, 2, 3, \ldots\ldots, n$.

Ex. 1. Expand $(\frac{1}{3}a-3b)^6$.

Sol. $(\frac{1}{3}a-3b)^6={}^6C_0(\frac{1}{3}a)^6+{}^6C_1(\frac{1}{3}a)^5(-3b)$
$+{}^6C_2(\frac{1}{3}a)^4(-3b)^2+{}^6C_3(\frac{1}{3}a)^3(-3b)^3+{}^6C_4(\frac{1}{3}a)^2(-3b)^4$
$+{}^6C_5(\frac{1}{3}a)(-3b)^5+{}^6C_6(-3b)^6$.

$$=\frac{a^6}{729}-\frac{2}{27}a^5b+\frac{5}{3}a^4b^2-20a^3b^3+135a^2b^4-486ab^5+729b^6.$$

Ex. 2. Expand $(1-x+x^2)^4$.

Sol. $(1-x+x^2)^4$.

$={}^4C_0(1-x)^4+{}^4C_1(1-x)^3x^2+{}^4C_2(1-x)^2(x^2)^2+{}^4C_3(1-x)(x^2)^3+{}^4C_4(x^2)^4$

$=(1-x)^4+4x^2(1-x)^3+6x^4(1-x)^2+4x^6(1-x)+x^8$

$=({}^4C_0-{}^4C_1.x+{}^4C_2x^2-{}^4C_3x^3+{}^4C_4x^4)+4x^2({}^3C_0-{}^3C_1x+{}^3C_2x^2-{}^3C_3x^3)+6x^4(1-2x+x^2)+4x^6-4x^7+x^8$

$=1-4x+6x^2-4x^3+x^4+4x^2-12x^3+12x^4-4x^5+6x^4-12x^5+6x^6+4x^6-4x^7+x^8$

$=1-4x+6x^2-12x^3+19x^4-16x^5+10x^6-4x^7+x^8$.

Ex. 3. Find the general term in the expansion of

$$\left(x^3+\frac{1}{x^2}\right)^{12}.$$

Sol. General term $=T_{r+1}={}^{12}C_r.(x^3)^{12-r}\left(\frac{1}{x^2}\right)^r$

$$={}^{12}C_r\, x^{36-3r}.\frac{1}{x^{2r}}$$

$$={}^{12}C_r.x^{36-5r}.$$

Ex. 4. Find the co-efficient of x^6 in the expansion of

$$\left(x^3+\frac{1}{2x^2}\right)^{12}.$$

Sol. Suppose x^6 occurs in T_{r+1}.

Now $T_{r+1}={}^{12}C_r.(x^3)^{12-r}\left(\frac{1}{2x^2}\right)^r$

$$={}^{12}C_r.x^{36-3r}\frac{1}{2^r x^{2r}}={}^{12}C_r\frac{1}{2^r}\,x^{36-5r}.$$

$\therefore$ $36-5r=6$ or $5r=30$, *i e*, $r=6$.

$\therefore$ x^6 occurs in T_7 and the required co-efficient is ${}^{12}C_6 \dfrac{1}{2^6}$

$$=\frac{12\,!}{6\,!\,6\,!}\cdot\frac{1}{2^6}=\frac{12.11.10.9.8.7}{2.1.2.3.4.5.6}=\frac{231}{16}.$$

Ex. 5. Find the co-efficient of x^3 in the expansion of

$$\left(x^2+\frac{1}{3x^3}\right)^{10}.$$

Sol. Let x^3 occur in T_{r+1}.

Now $T_{r+1}={}^{10}C_r(x^2)^{10-r}\left(\dfrac{1}{3x^3}\right)^r$.

$$={}^{10}C_r.\frac{1}{3^r}\cdot x^{20-5r}$$

$\therefore$ $20-5r=3$, *i.e.*, $r=\frac{17}{5}$, which is impossible, since r must be a positive integer.

Ex. 6. Find the term independent of x in the expansion of $\left(2x^4-\dfrac{1}{3x^7}\right)^{11}$.

Sol. Let T_{r+1} be the term independent of x.

Now $T_{r+1}={}^{11}C_r.(2x^4)^{11-r}.\left(-\dfrac{1}{3x^7}\right)^r$

$$=2^{11-r}.(-\tfrac{1}{3})^r.{}^{11}C_r\cdot x^{44-11r}$$

$\therefore$ $44-11r=0$ or $r=4$.

$\therefore$ $T_{4+1}=T_5$ is the term independent of x and it is

$$=2^{11-4}(\tfrac{1}{3})^4.{}^{11}C_4.$$

$$=2^7.\frac{1}{3^4}\cdot\frac{11.10.9.8}{1.2.3.4}=\frac{14080}{27}.$$

Ex. 7. Find the middle term in the expansion of

$$\left(x^2+\frac{1}{8}\right)^8.$$

Sol. There are 9 terms in the expansion, so that T_5 is the middle term.

Now $T_{r+1}={}^8C_r(x^2)^{8-r}\left(\dfrac{1}{x}\right)^r$

$$={}^8C_r x^{16-3r},$$

$\therefore$ Putting $r=4$, the required term

$$=T_5={}^8C_4 . x^{16-12}=\frac{8.7.6.5}{1.2.3.4}x^4=70x^4.$$

Ex. 8 Find the two middle terms in the expansion of

$$\left(x^3+\frac{1}{x^2}\right)^7.$$

Sol. There are $7+1=8$ terms in the expansion, so that T_4 and T_5 are the two middle terms.

$$\text{Now } T_{r+1}={}^7C_r(x^3)^{7-r}.\left(\frac{1}{x^2}\right)^r$$

$$={}^7C_r x^{21-5r}.$$

$\therefore$ putting $r=3$ and 4 respectively,

$$T_4={}^7C_3 x^{21-15}=35x^6$$

$$\text{and } T_5={}^7C_4 x^{21-20}=35x.$$

Note.— If p is the number of terms in a series then

(*i*) when p is odd, the series has one middle term, *viz.*, $T_{\frac{p+1}{2}}$

(*ii*) when p is even, the series has two middle terms, *viz.*, $T_{\frac{p}{2}}$, $T_{\frac{p}{2}+1}$.

Ex. 9. Show that the middle term in the expansion of $(1+x)^{2n}$, n being a positive integer is $\dfrac{1.3.5...(2n-1)}{n\,!}.2^n x^n$.

(*P. U.* 1942)

Sol. Number of terms in the expansion is $2n+1$, which is odd. Hence there is one middle term, *viz.*, T_{n+1}.

$$\therefore \text{ The required term}={}^{2n}C_n . x^n=\frac{2n\,!}{(n\,!)^2}x^n$$

$$=\frac{1.2.3.4...(2n-1).2n}{(1.2.3..n)^2}x^n$$

$$=\frac{1.3.5...(2n-1)}{n\,!}.\frac{2.4.6...2n}{1.2.3...n}x^n$$

$$=\frac{1.3.5...(2n-1)}{n\,!}.2^n x^n.$$

Exercises

1. Expand

(i) $\left(\sqrt{x}+\dfrac{1}{2\sqrt{x}}\right)^5$. (ii) $\left(x^{\frac{1}{3}}-\dfrac{1}{3x^{\frac{2}{3}}}\right)^9$,

(iii) $\left(1+\dfrac{1}{\sqrt{x}}\right)^8$, (iv) $\left(\dfrac{x}{y}-\dfrac{y}{x}\right)^6$.

2. Expand in powers of x

(i) $(1+x+x^2)^3$, (ii) $(1-x^2+x^4)^3$.

3. Apply the Binomial Theorem to evaluate

(i) $(1001)^4$, (ii) 998^4, (iii) 999^5. (*P. U.* 1932)

4. Simplify (i) $(x+y)^5+(x-y)^5$

(ii) $(x+y)^6-(x-y)^6$

(iii) $(x+\sqrt{x^2-y^2})^7-(x-\sqrt{x^2-y^2})^7$.

5. Put down the general terms in the following expansions : —

(i) $\left(2x^2-\dfrac{3}{x}\right)^{10}$, (ii) $(5a^2+8x)^{25}$,

(iii) $(ax^p+bx^{-q})^n$ (iv) $\left(x+\dfrac{1}{x^3}\right)^{4n}$,

(v) $\left(2x+\dfrac{1}{3x^2}\right)^9$, (vi) $\left(x-\dfrac{1}{x}\right)^{8n}$,

(vii) $\left(x^2+\dfrac{1}{x^2}\right)^{3n}$. (viii) $\left(2x^3-\dfrac{1}{x}\right)^{12}$,

(ix) $\left(\dfrac{x}{y}+\dfrac{y}{x}\right)^{2n+1}$, (x) $\left(x+\dfrac{1}{x}\right)^{10}$.

6. Find the co-efficient of

(i) x^5 in $\left(2x^2-\dfrac{3}{x}\right)^{10}$ (*P. U.*)

(ii) x^{16} in $(2x^2-x)^{10}$ (*C. U.* 1942)

(iii) x^{11} in $(5a^2+8x)^{25}$. (*P.U.*)

7. In the expansion of $(1+x)^{m+n}$, where m and n are positive integers, prove that co-efficient of x^m and x^n are equal. (*C. U.* 1932)

8. Prove that the co-efficient of x^n in the expansion of $(1+x)^{2n}$ is double the co-efficient of x^n in the expansion of $(1+x)^{2n-1}$. (*P. U.* 1920)

9. Find the $(n+1)$th term from the end in the expansion of $\left(x-\dfrac{1}{x}\right)^{3n}$. (*P. U.* 1939)

10. If the co-efficients of x^4 and x^6 in $(1+2x)^n$ are 1120 and 1792 respectively, find the value of n. (*P. U.*)

11. Find the term independent of x in

(*i*) $\left(2x^2-\dfrac{1}{x}\right)^{12}$ (*P.U.*) (*ii*) $\left(2x+\dfrac{1}{3x^2}\right)^9$ (*C. U.* 1936)

(*iii*) $\left(x-\dfrac{1}{x}\right)^{4p}$ (*iv*) $\left(x-\dfrac{1}{x^2}\right)^{4n}$ (*P. U.*)

(*v*) $\left(x^2+\dfrac{1}{x^2}\right)^{3n}$ (*vi*) $\left(\frac{3}{2}x^2-\dfrac{1}{3x}\right)^9$ (*P. U.* 1940)

12. Find the term independent of x in

(*i*) $\left(1+x\right)^5\left(1+\dfrac{1}{x}\right)^7$, (*ii*) $\left(1+x^2\right)^3\left(1+\dfrac{1}{x^2}\right)^4$,

(*iii*) $\left(1+x\right)^m\left(1+\dfrac{1}{x}\right)^n$, where m, n are positive integers.

23. Find the middle term (or terms) in the following :—

(*i*) $(1+x)^{20}$ (*ii*) $(a^2+x^2)^{11}$

(*iii*) $\left(x+\dfrac{1}{x}\right)^{2n}$ (*P.U.* 1909 (*iv*) $\left(3a-\dfrac{a^3}{b}\right)^9$ (*P. U.* 1912)

(*v*) $\left(x-\dfrac{1}{x}\right)^{10}$ (*vi*) $\left(ax-\dfrac{b}{x^2}\right)^{2p+1}$

(*vii*) $(1+x)^{100}$. (*C. U.* 1922)

14. Prove that the co-efficient of the middle term in the expansion of $(1+x)^2$ is equal to the sum of the co-efficients of the two middle terms in the expansion of $(1+x)^{2n-1}$.

15. Find out the co-efficient of two middle terms in $(1+x)^{2n+1}$ and the co-efficient of the middle term in $(1+x)^{2n}$. Prove that the quotient of the first by the second always lies between $\frac{3}{2}$ and 2. (*P. U.* 1937)

Binomial Co-efficients

9·3. The successive co-efficients in the expansion of $(a+x)^n$, *viz.*, ${}^nC_0, {}^nC_1, {}^nC_2 \ldots\ldots, {}^nC_n$ are called **Binomial Co-efficients**. The binomial co-efficients are sometimes denoted by $C_0, C_1, C_2\ldots\ldots C_n$.

The following properties may be noted :—

(*i*) *Co-efficients equidistant from the beginning and end are equal.*

This is so because ${}^nC_r = {}^nC_{n-r}$.

(*ii*) *Sum of the Binomial co-efficients*$=2^n$.

For $(a+x)^n = C_0a^n + C_1a^{n-1}x + C_2a^{n-2}x^2 + \ldots\ldots$

$$\ldots\ldots + C_{n-1}ax^{n-1} + C_nx^n.$$

Putting $a=x=1$, we get $2^n = C_0 + C_1 + C_2 + \ldots + C_{n-1} + C_n$.

(*iii*) *Sum of the odd co-efficients*=*sum of even co-efficients.*

For $(a-x)^n = C_0a^n - C_1a^{n-1}x + C_2a^{n-2}x^2 - C_3a^{n-3}x^3 + \ldots$

Putting $a=x=1$, we get

$$0 = C_0 - C_1 + C_2 - C_3 + \ldots\ldots$$

$$\therefore \quad C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots$$

Ex. 10. Find the value of

$$C_0 + \tfrac{1}{2}.C_1 + \tfrac{1}{3}.C_2 + \tfrac{1}{4}.C_3 + \ldots + \frac{1}{n+1}.C_n.$$

Sol. The given expression

$$= 1 + \tfrac{1}{2}.n + \tfrac{1}{3}.\frac{n(n-1)}{1.2} + \tfrac{1}{4}.\frac{n(n-1)(n-2)}{1.2.3} + \ldots + \frac{1}{n+1}.1$$

$$= \frac{1}{n+1}\Big[(n+1) + \frac{(n+1)n}{1.2} + \frac{(n+1)n(n-1)}{1.2.3}$$

$$+ \frac{(n+1)n(n-1)(n-3)}{1.2.3.4} + \ldots + 1\Big]$$

$$= \frac{1}{n+1}[{}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \ldots + {}^{n+1}C_{n+1}]$$

$$= \frac{1}{n+1}[({}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + \ldots + {}^{n+1}C_{n+1}) - {}^{n+1}C_0]$$

$$= \frac{1}{n+1}[2^{n+1} - 1] = \frac{2^{n+1}-1}{n+1}.$$

Ex. 11. Find the value of

$$C_0C_1 + C_1C_2 + C_2C_3 + \ldots + C_{n-1}C_n$$

Sol. We know

$$(1+x)^n=C_0+C_1x+C_2x^2+C_3x^3+\ldots+C_{n-3}x^{n-3}$$
$$+C_{n-2}x^{n-2}+C_{n-1}x^{n-1}+C_nx^n \quad \ldots(i)$$

Also $(1+x)^n=C_n+C_{n-1}x+C_{n-2}x^2+C_{n-3}x^3+\ldots$

$$\ldots+C_3x^{n-3}+C_2x^{n-2}+C_1x^{n-1}+C_0x^n \quad \ldots(ii)$$
$$[\because \; C_0=C_n,\; C_1=C_{n-1},\; C_2=C_{n-2} \text{ and so on})$$

Now multiplying the right hand side of (i) and (ii) we see that the expression to be evaluated is the co-efficient of x^{n-1} in this product. Also the product of the left hand sides is $(1+n)^{2n}$.

$$\therefore \quad C_0C_1+C_1C_2+C_2C_3+\ldots+C_{n-1}C_n$$
$$=\text{co-efficient of } x^{n-1} \text{ in } (1+n)^{2n}$$
$$={}^{2n}C_{n-1}=\frac{2n\,!}{(n-1)\,!\,(n+1)\,!}.$$

Exercises

1. ${}^{25}C_0+{}^{25}C_1+{}^{25}C_2+\ldots{}^{25}C_{25}$.
2. ${}^{12}C_1+{}^{12}C_2+{}^{12}C_3+\ldots+{}^{12}C_{12}$.
3. ${}^{14}C_0+{}^{14}C_1+{}^{14}C_2+\ldots+{}^{14}C_{13}$.
4. ${}^{16}C_1+{}^{16}C_2+{}^{16}C_3+\ldots+{}^{16}C_{15}$.
5. ${}^{20}C_2+{}^{20}C_3+{}^{20}C_4+\ldots+{}^{20}C_{18}$.

If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_{n-1}x^{-1}+C_nx_n$, prove that

6. $C_1+2\,C_2+3.C_3+\ldots+n.C_n=n.2^{n-1}$. (*C. U.* 1938)
7. $(C_0+C_1)(C_1+C_2)\ldots(C_{n-1}+C_n)$
$$=C_1C_2C_3\ldots\ldots C_n\,\frac{(n+1)^n}{n\,!}.$$
8. $C_1-2.C_2+3.C_3-\ldots+(-1)^{n-1}.n.C_n=0$.
9. $C_0^2+C_1^2+C_2^2+\ldots+C_n^2=\dfrac{2n\,!}{(n\,!)^2}$.
10. $C_0C_2+C_1C_3+C_2C_4+\ldots+C_{n-2}C_n=\dfrac{2n\,!}{(n-2)\,!\,(n+2)\,!}$
11. $C_0C_5+C_1C_6+C_2C_7+\ldots+C_{n-5}C_n=\dfrac{2n\,!}{(n-5)\,!\,(n+5)\,!}$.

12. $(C_0+C_1+C_2+\ldots+C_n)^2={}^{2n}C_0+{}^{2n}C_1+{}^{2n}C_2+\ldots$
$\ldots\ldots+{}^{2n}C_{2n}.$

9·4. Miscellaneous Examples.

Ex. 12. If three successive co-efficients in the expansion of $(1+x)^n$ are 220, 495, 792, find n. *(P. U. 1938)*

Sol. Let the given numbers be the co-efficients of T_r, T_{r+1}, T_{r+2}. Now

$$T_r={}^nC_{r-1}.x^{r-1},\ T_{r+1}={}^nC_r.x^r,\ T_{r+2}={}^nC_{r+1}.x^{r+1}$$

$\therefore\ {}^nC_{r-1}=220,\ {}^nC_r=495,\ C_{r+1}=792$

$\therefore\ {}^nC_r : {}^nC_{r-1}=495 : 220$

$$i.\,e.,\ \frac{n\,!}{r\,!\,(n-r)\,!}\div\frac{n\,!}{(r-1)\,!\,(n-r+1)\,!}=\frac{495}{220}$$

$$i.\,e.,\ \frac{n-r+1}{r}=\frac{495}{220}=\frac{9}{4}\ \text{ or }\ 4n=13r-4 \qquad \ldots\ldots(i)$$

Similarly ${}^nC_{r+1} : {}^nC_r=792 : 495$

$$i.\,e.,\ \frac{n\,!}{(r+1)\,!\,(n-r+1)\,!}\div\frac{n\,!}{r\,!\,(n-r)\,!}=\frac{792}{495}=\frac{8}{5}$$

$$i.\,e.,\ \frac{n-r}{r+1}=\frac{8}{5}\ \text{ or }\ 5n=13r+8 \qquad \ldots\ldots(ii)$$

From (i) and (ii) by subtraction $n=12$.

Ex. 13. Find the term with the numerically greatest co-efficient in the expansion of $(1+x)^{16}$.

Sol. The co-efficient of T_{r+1} is $\gtrless$ that of T_r

according as ${}^{16}C_r \gtrless {}^{16}C_{r-1}$

$$i.\,e.,\ \text{according as } \frac{16\,!}{r\,!\,(16-r)\,!}\gtrless\frac{16\,!}{(r-1)\,!\,(17-r)\,!}$$

$i.\,e.$, according as $17-r \gtrless r$ $i.\,e.$, according as $\frac{17}{2}\gtrless r$

Hence putting $r=1, 2, 3\ldots 8$, we see that the co-efficients of $T_1, T_2, T_3\ldots, T_9$ are in ascending order of magnitude.

Again, putting $r=9, 10, 11\ldots$, we see that the co-efficients of $T_9, T_{10}, T_{11}\ldots$ are in descending order of magnitude.

Therefore T_9 is the term with numerically greatest co-efficient.

Ex. 14. Find the term or terms with the numerically greatest co-efficient in the expansion of $(1+x)^{11}$.

Sol. As in the above example, the co-efficient of $T_{r+1} \gtreqless$ that of T_r,

$$\text{according as } {}^{11}C_r \gtrless {}^{11}C_{r-1}$$

i.e., $$\text{according as } \frac{11\,!}{r\,!\,(11-r)\,!} \gtrless \frac{11\,!}{(r-1)\,!\,(12-r)\,!}$$

i.e., $$\text{according as } r \gtrless 6.$$

$\therefore$ Putting $r=1, 2, 3, 4, 5$, we see that the co-efficients of $T_1, T_2, T_3, \ldots T_6$ are in ascending order of magnitude.

Putting $r=6$, we see that the co-efficients of T_6 and T_7 are equal.

Putting $r=7, 8, 9, \ldots$ we see that the co-efficients of T_7, T_8, T_9, $T_{10} \ldots$ are in descending order of magnitude.

Hence the co-efficients of T_6 and T_7 are numerically the greatest.

Ex, 15. Find the numerically greatest term or terms in the expansion of $(5+x)^{21}$, when $x=\frac{1}{2}$.

Sol. We have $T_{r+1} = {}^{21}C_r . 5^{21-r} . x^r$

and $$T_r = {}^{21}C_{r-1} . 5^{22-r} . x^{r-1}$$

$\therefore$ $T_{r+1} \gtrless T_r$, according as

$${}^{21}C_r \; 5^{21-r} \; x^r \gtrless {}^{21}C_{r-1} \; 5^{22-r} \; x^{r-1}$$

i.e., $$\frac{21\,!}{r\,!\,(21-r)\,!} . 5^{21-r} \, x^{r-1} . x \gtrless \frac{21\,!}{(r-1)\,!\,(22-r)\,!} . 5^{21-r} . 5 . x^{r-1}$$

i. e., $$\frac{x}{r} \gtrless \frac{5}{22-r} \text{ or putting } x=\tfrac{1}{2}$$

$$\text{according as } 22-r \gtrless 10r$$

i.e., $$\text{according as } 2 \gtrless r.$$

$\therefore$ putting $r=1$, we see that $T_2 > T_1$

putting $r=2$, we see that $T_3 = T_2$

and putting $r=3, 4, 5, \ldots$ we see that

$T_3, T_4, T_5, \ldots\ldots$ are in descending order of magnitude. Hence second and third terms are equal and numerically the greatest.

Ex. 16. Find the numerically greatest term or terms in the expansion of $(1+x)^{13}$ when $x=\frac{2}{3}$.

Sol. As in the above example

$$T_{r+1} \gtrless T_r \text{ according as}$$

$$^{13}C_r . x^r \gtrless {}^{13}C_{r-1}\, x^{r-1}$$

i.e., $\quad (14-r)x \gtrless r$; *i. e.*, $(14-r)\,\frac{2}{3} \gtrless r$

i.e., $\quad 28 \gtrless 5r$ or $5\frac{3}{5} \gtrless r$

$\therefore$ putting $r=1, 2, 3, 4, 5$, we see that

$$T_1, T_2, T_3, T_4, T_5, T_6$$

are in ascending order of magnitude, while putting $r=6, 7, 8\ldots$, we see that

$$T_6, T_7, T_8, \ldots\ldots$$

are in descending order of magnitude.

Hence T_6 is the numerically greatest term in the expansion.

Exercises

1. Find the greatest term in the expansion of $(2+5x)^{10}$ when $x=\frac{1}{3}$. (*Allahabad* 1920)

2. Find the greatest term in the expansion of $\left(3+\frac{1}{x}\right)^{13}$ when $x=\frac{2}{5}$.

3. Find the numerically greatest term in the expansions of

(*i*) $(5+7x)^{15}$ when $x=\frac{1}{3}$

(*ii*) $\left(7-\frac{2}{x}\right)^{11}$ when $x=\frac{1}{5}$

(*iii*) $\left(11-\frac{x}{13}\right)^{100}$ when $x=\frac{1}{7}$.

4. If the 6th, 7th, 8th, and 9th terms in the expansion of $(x+y)^n$ be p, q, r, s respectively, prove that

$$3r(q^2-pr)=4p(r^2-qs).$$

5. Determine the binomial expression of which 4 consecutive terms are 576, 2160, 4320, 4860. (*P. U.* 1931)

CHAPTER X

Binomial Theorem—Any Index

10·1. In the last chapter we obtained the result

$$(1+x)^n=1+nx+\frac{n(n-1)}{2\,!}x^2+\frac{n(n-1)(n-2)}{3\,!}x^3+\ldots\ldots$$

for a positive integer n, where no restriction was imposed on the value of x.

We also know that

$$T_{r+1}=\frac{n(n-1)(n-2)\ldots(n-r+1)}{r\,!}x^r.$$

Putting $r=n$, we get

$$T_{n+1}=\frac{n(n-1)(n-2)\ldots(n-n+1)}{n\,!}x^n=x^n.$$

If we put $r=n+1$, we obtain

$$T_{n+2}=\frac{n(n-1)(n-2)\ldots(n-n-1+1)}{(n+1)\,!}x^{n+1}=0.$$

Every term beyond T_{n+2} will also be zero.

Thus if n is a positive integer, the number of terms in the above expansion is $n+1$ and the last term is x^n.

The expansion has been mathematically proved in the last chapter to be true for integral index. If, however; n is not a positive integer, and is a negative integer or a fraction (positive or negative) and we suppose the expansion still to be true, then it is not difficult to see that no term in the expansion will be zero. For in the expansion for T_{r+1}, the successive factors being n, $n-1$, $n-2$,……, none of them will be zero for any integral value of r, if n is not a positive integer. Therefore, the expansion, if at all true for a negative integer or a fraction (positive or nagative) will form an infinite series.

10·2. The proof of the Binomial Theorem for the case where n is a negative integer or a fraction (positive or negative) is a little too difficult to be given at this stage, and is beyond the scope of the present work. It is proved in books of Algebra that the expansion

$$(1+x)^n=1+nx+\frac{n(n-1)}{2\,!}x+\frac{n(n-1)(n-2)}{3\,!}x^3+\ldots\ldots$$
$$\ldots\ldots+\frac{n(n-1)(n-2)\ldots(n-r+1)}{r\,!}x^r+\ldots\ldots\text{to }\infty$$

is true for a negative integer or a fraction n, provided x is numerically less than 1.

For our purposes we shall assume this result without any proof and apply it in the solution of examples. That the result is not true for x numerically greater than 1, can be easily seen by putting $x=-2$ and $n=-1$. Thus we get

$$(1-2)^{-1}=1+(-1)(-2)+\frac{(-1)(-1-1)}{2\,!}(-2)^2$$
$$+\frac{(-1)(-1-1)(-1-2)}{3\,!}(-2)^3+\ldots\ldots\text{to }\infty$$

which gives

$$-1=1+2+4+8+\ldots\ldots\text{to }\infty$$

an obviously absurd result.

Ex. 1. Find the general term in the expansions of

(*i*) $(1-x)^{-1}$ (*ii*) $(1+x)^{-2}$ (*iii*) $(1-x)^{-3}$

(*iv*) $(1-2x)^{-\frac{3}{2}}$. (*P. U.* 1916)

Sol. (*i*) Here $n=-1$; therefore

$$T_{r+1}=\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}(-x)^r$$
$$=\frac{(-1)(-1-1)(-1-2)\ldots(-1-r+1)}{1.2.3\ldots r}(-x)^r$$
$$=(-1)^r.\frac{1.2.3\ldots r}{1.2.3\ldots r}(-1)^r.x^r$$
$$=(-1)^{2r}.x^r=x^r.$$

(*ii*) Here $T_{r+1}=\dfrac{-2(-2-1)(-2-2)\ldots(-2-r+1)}{1.2.3\ldots r}x^r$

$$=(-1)^r.\frac{2.3.4\ldots(r+1)}{1.2.3\ldots r}x^r$$
$$=(-1)^r(r+1)x^r.$$

(*iii*) Here $T_{r+1}=\dfrac{-3(-3-1)(-3-2)\ldots(-3-r+1)}{1.2.3\ldots r}(-x)^r$

$$=(-1)^r\frac{3.4.5\ldots(r+2)}{1.2.3\ldots r}(-1)^r x^r$$

$$=(-1)^{2r}\frac{(r+1)(r+2)}{1.2}x^r$$

$$=\frac{(r+1)(r+2)}{1.2}x^r.$$

(*iv*) Here $T_{r+1}=\dfrac{-\frac{3}{2}(-\frac{3}{2}-1)(-\frac{3}{2}-2)\ldots(-\frac{3}{2}-r+1)}{1.2.3\ldots r}(-2x)^r$

$$=(-1)^r\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\frac{2r+1}{2}}{1.2.3\ldots r}(-1)^r.2^r.x^r$$

$$=(-1)^{2r}.\frac{3.5.7\ldots(2r+1)}{1.2.3\ldots r}\;\frac{2^r}{2^r}x^r$$

$$=\frac{3.5.7\ldots(2r+1)}{1.2.3\ldots r}.x^r.$$

Ex. 2. Expand to 4 terms

(*i*) $(x+y)^{-\frac{1}{2}}$ (*ii*) $(x^{-\frac{1}{2}}+y^{-\frac{1}{2}})^{-\frac{1}{3}}$

(*iii*) $\left(2x^3-\dfrac{7y}{x}\right)^{-3}$

and state in each case the condition for the expansion to be valid.

Sol. $(x+y)^{-\frac{1}{2}}=x^{-\frac{1}{2}}\left(1+\dfrac{y}{x}\right)^{-\frac{1}{2}}$

$$=x^{-\frac{1}{2}}\Big[1+(-\tfrac{1}{2})\frac{y}{x}+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{1.2}\;\frac{y^2}{x^2}$$

$$+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1.2.3}\;\frac{y^3}{x^3}+\ldots\ldots\Big]$$

$$=x^{-\frac{1}{2}}\left[1-\frac{y}{2x}+\frac{3y^2}{8x^2}-\frac{5}{16}\cdot\frac{y^3}{x^3}+\ldots\ldots\right]$$

$$=\frac{1}{x^{\frac{1}{2}}}-\frac{1}{2}\cdot\frac{y}{x^{\frac{3}{2}}}+\frac{3}{8}\cdot\frac{y^2}{x^{\frac{5}{2}}}-\frac{5}{16}\cdot\frac{y^3}{x^{\frac{7}{2}}}+\ldots$$

The expansion in this case is valid, if y/x is numerically less than 1, *i. e.*, y is numerically less than x.

(*ii*) $(x^{-\frac{1}{2}}+y^{-\frac{1}{2}})^{-\frac{1}{3}}=(x^{-\frac{1}{2}})^{-\frac{1}{3}}\left(1+\dfrac{y}{x^{-\frac{1}{2}}}\right)^{-\frac{1}{3}}=x^{\frac{1}{6}}\left(1+\dfrac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}\right)^{-\frac{1}{3}}$

$$=x^{\frac{1}{8}}\left[1+(-\tfrac{1}{3})\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}+\frac{(-\frac{1}{3})(-\frac{1}{3}-1)}{1.2}\cdot\left(\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}\right)^2\right.$$

$$\left.+\frac{(-\frac{1}{3})(-\frac{1}{3}-1)(-\frac{1}{3}-2)}{1.2.3}\left(\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}\right)^2+\ldots\ldots\right]$$

$$=x^{\frac{1}{8}}\left[1-\tfrac{1}{3}.\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}+\frac{2}{9}\cdot\frac{x}{y}-\frac{14}{81}\cdot\frac{x^{\frac{3}{2}}}{y^{\frac{3}{2}}}+\ldots\right]$$

$$=x^{\frac{1}{8}}-\tfrac{1}{3}.\frac{x^{\frac{2}{3}}}{y^{\frac{1}{2}}}+\frac{2}{9}\frac{x^{\frac{7}{8}}}{y}-\frac{14}{81}\cdot\frac{x^{\frac{5}{8}}}{y^{\frac{3}{2}}}+\ldots\ldots$$

In this case the expansion is valid, if $\dfrac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}$ is numerically less than 1 *i.e.*, $x/y<1$ or $x<y$.

(*iii*) $\left(2x^3-\frac{7y}{x}\right)^{-3}=(2x^3)^{-3}\left(1-\frac{7y}{2x^4}\right)^{-3}=\frac{1}{8x^9}\cdot\left(1-\frac{7y}{2x^4}\right)^{-3}$

$$=\frac{1}{8x^9}\left[1+(-3)\left(-\frac{7}{2}\cdot\frac{y}{x^4}\right)+\frac{(-3)(-3-1)}{1.2}\cdot\left(-\frac{7}{2}\cdot\frac{y}{x^4}\right)^2\right.$$

$$\left.+\frac{(-3)(-3-1)(-3-2)}{1.2.3}\left(\frac{7}{2}\cdot\frac{y}{x^4}\right)^3+\ldots\ldots\ldots\right]$$

$$=\frac{1}{8x^9}\left[1+\frac{21}{2}\cdot\frac{y}{x^4}+\frac{147}{2}\cdot\frac{y^2}{x^8}+\frac{1715}{4}\cdot\frac{y^3}{x^{12}}+\ldots\ldots\right]$$

$$=\frac{1}{8}\cdot\frac{1}{x^9}+\frac{21}{16}\cdot\frac{y}{x^{13}}+\frac{147}{16}\cdot\frac{y^2}{x^{17}}+\frac{1715}{32}\cdot\frac{y^3}{x^{21}}+\ldots\ldots$$

10·3. The general term in the expansion of $(1+x)^n$, where x and n are both positive is given by

$$T_{r+1}=\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}x^r.$$

The factors n, $n-1$, $n-2$,...are in descending order of magnitude and if n is not a positive integer, there will be a value of r, r' say, such that the factor $n-r'+1$ and all those that follow it are negative, whereas $n-1$, $n-2$, $n-3$,...$n-r'+2$ are all positive.

Hence in the expansion of $(1+x)^n$, where x and n are both positive, the terms are all alternately negative and positive. It is sometimes required to find out the first negative term in the expansion of $(1+x)^n$. The procedure to be adopted is illustrated below :—

Ex. 3. Find the first negative term in the expansion of

(*i*) $(1+x)^{\frac{7}{2}}$, (*ii*) $(1+x)^{\frac{11}{17}}$.

Sol. (*i*) Here $T_{r+1}=\dfrac{\frac{7}{2}\cdot(\frac{7}{2}-1)(\frac{7}{2}-2)\ldots(\frac{7}{2}-r+1)}{1.2.3\ldots r}\cdot x^r$

$$=\frac{7.5.3\ldots(11-2r)(9-2r)}{2^r\, r\,!}\cdot x^r.$$

$\therefore$ T_{r+1} is the first negative term, provided

$$9-2r<0 \text{ and } 11-2r>0.$$

Therefore $r=5$ and T_6 is the first negative term.

(*ii*) T_{r+1} will be the first negative, term if

$$\tfrac{11}{17}-r+1<0 \text{ and } \tfrac{11}{17}-r+2>0$$

Hence $r=2$ so that T_3 is the first negative term.

10·4. The general term in the expansion of $(1+x)^{-n}$, where x and n are both positive is

$$T_{r+1}=\frac{-n(-n-1)(-n-2)\ldots(-n-r+1)}{1.2.3\ldots r}x^r$$

$$=(-1)^r\cdot\frac{n(n+1)(n+2)\ldots(n+r-1)}{1.2.3\ldots r}\cdot x^r.$$

Here the factors $n, n+1, n+2, \ldots$ are in ascending order of magnitude and since n is positive, these factors are all positive. Therefore the sign of T_{r+1} is the same as that of $(-1)^r$. In other words, in the expansion of $(1+x)^{-n}$, where x and n are both positive, the terms are alternately positive and negative.

10·5. The general term in the expansion of $(1-x)^n$, where x and n are both positive is

$$T_{r+1}=\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots. r}(-x)^r$$

$$=(-1)^r\cdot\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}\cdot x^r.$$

As in 10·2 there is a value of r, say r', such that n, $n-1, \ldots\ldots n-r'+2$ are all positive whereas $n-r'+1$, $n-r'$,

$n-r'-1,\ldots$are all negative. Hence in this case, for $0 \leqslant r \leqslant r'-1$, T_{r+1} has the same sign as $(-1)^r$, *i.e.*, $T_1, T_2, \ldots T_r$, are alternately positive and negative. For $r > r'-1$, T_r has r factors n, $n-1$, $n-2,\ldots$in the numerator, out of which $r'-1$ are positive and $r-r'+1$ are negative. Hence for $r > r'-1$, T_{r+1} has the same sign as that of $(-1)^r (-1)^{r-r'+1} = (-1)^{1-r'}$ which does not depend upon r. This shows that $T_{r'+1}$, $T_{r'+2}$, $T_{r'+3}$,...have the same sign, being all positive or all negative according as r' is odd or even.

Ex. 4. In the expansion of $(1-x)^{\frac{17}{4}}$, where $x > 0$, find after which term, the terms are constantly of the same sign. State also whether they are constantly positive or negative.

Sol. Here $$T_{r+1} = \frac{\frac{17}{4}.(\frac{17}{4}-1)\ldots(\frac{17}{4}-r+2)(\frac{17}{4}-r+1)}{1.2.4\ldots\ldots r}(-x)^r$$

$$=(-1)^r \frac{\frac{17}{4}(\frac{17}{4}-1)\ldots(\frac{17}{4}-r+2)(\frac{17}{4}-r+1)}{1.2.3\ldots\ldots r} x^r.$$

Therefore the terms T_{r+1} are alternately positive and negative, where $0 \leqslant r \leqslant r'-1$ and r' is given by

$$\tfrac{7}{4}-r'+1<0 \text{ and } \tfrac{7}{4}-r'+2<0.$$

Hence $r'=3$ and the terms $T_3, T_4, T_5 \ldots\ldots$are all of the same sign. Also sign of T_3 is positive, therefore the constant sign is positive.

10·6. The general term in the expansion of $(1-x)^{-n}$ where x and n are both positive, is

$$T_{r+1} = \frac{-n(-n-1)(-n-2)\ldots(-n-r+1)}{1.2.3\ldots\ldots r}.(-x)^r$$

$$=(-1)^r . \frac{n(n+1)(n+2)\ldots(n+r-1)}{r\,!}(-1)^r x^r$$

$$=(-1)^{2r} \frac{n(n+1)(n+2)\ldots(n+r-1)}{r\,!} x^r$$

$$=\frac{n(n+1)(n+2)\ldots(n+r-1)}{r\,!} x^r$$

Now the factors n, $n+1$, $n+2$...are all in ascending order of magnitude and since n, the least of them is positive, all of them are positive. *Hence in the expansion of* $(1-x)^{-n}$, *where* $x > 0$, $n > 0$, *all the terms are positive.*

As particular cases the student is advised to remember the following expressions :—

$$(1-x)^{-1}=1+x+x^2+x^3+\ldots x^r+\ldots$$
$$(1-x)^{-2}=1+2x+3x^2+4x^3+\ldots+(r+1)x^r+\ldots$$
$$(1-x)^{-3}=1+3x+6x^2+10x^3+\ldots\ldots+\frac{(r+1)(r+2)}{2}x^r+\ldots$$
$$(1-x)^{-4}=1+4x+10x^2+20x^3+\ldots$$
$$+\frac{(r+1)(r+2)(r+3)}{1.2.3}x^r+\ldots$$

Exercises

1. Find the general term in the expansion of :—

(*i*) $(1-2x)^{-\frac{3}{2}}$. (*P. U.* 1916)

(*ii*) $(1+x)^{-\frac{p}{q}}$ (*iii*) $\dfrac{1}{\sqrt[4]{1-x}}$ (*iv*) $(1-px)^{\frac{1}{p}}$.

(*v*) $(1-4x)^{-\frac{3}{4}}$ (*vi*) $(1-x)^{-n}$. (*P. U.* 1942, 1943)

2. Expand to 4 terms :—

(*i*) $(a^2x^{-\frac{1}{2}}-a^{-\frac{2}{3}}x^{\frac{1}{2}})^{-4}$. (*P. U.* 1906)

(*ii*) $(a^{-\frac{1}{3}}+b^{-\frac{1}{3}})^{-3}$.

(*iii*) $\left(\dfrac{3}{a}-2x\right)^{-4}$ (*iv*) $(a+bx)^{-\frac{p}{q}}$. (*Allahabad* 1931)

(*v*) $(2-3x^2)^{-\frac{4}{5}}\left(x<\sqrt{\dfrac{2}{3}}\right)$. (*Allahabad* 1932)

(*vi*) $(256+64x)^{\frac{11}{2}}$.

3. Find the co-efficient of x^p in the expansion of
$\dfrac{(1+x)^2}{(1-x)^3}$. (*P. U.* 1919 *Supp.*)

4. Find the co-efficient of x^6 in the expansion of
$(1+x+x^2)^{-3}$. (*P. U.* 1919)

5. Expand $(1+4x+x^2)^{\frac{1}{2}}$ to x^3. For what value of x is this expansion valid ? (*P. U.* 1920)

6. Find the co-efficient of x^{10} in the expansion of

$$\frac{1+2x}{(1-2x)^3}, \text{ when } n<\tfrac{1}{2}.$$

7. Show that the co-efficient of x^n in the expansion of $\frac{(1+x)^2}{(1-x)^2}$ is $4n$. Point out an exceptional case, if any $(x<1)$.

(*P. U.* 1931)

8. Find which is the first negative term in the expansion of :—

(*i*) $(1+\frac{2}{3}x)^{\frac{3}{4}}$ (*P. U.* 1918) (*ii*) $(1+2x)^{\frac{7}{3}}$ (*P. U.* 1919)

(*iii*) $(1+\frac{2}{3}x)^{\frac{21}{4}}$. (*P. U.* 1942) (*iv*) $\left(1+\frac{x}{12}\right)^{\frac{11}{18}}$

(*v*) $(1+\frac{2}{3}x)^{\frac{200}{3}}$ (*vi*) $(1+\frac{2}{3})^{\frac{13}{4}}$. (*P. U.* 1918)

(*viii*) $(1+x)^{p+\frac{m}{n}}$, where p is a positive integer and $\frac{m}{n}$ a proper fraction.

9. Find the co-efficient of x^r in the expansion of $(1-4x)^{\frac{1}{2}}$.

10. Show that

$$(1+x)^n=2^n\left[1-n\,\frac{1-x}{1+x}+\frac{n(n+1)}{2\,!}\left(\frac{1-x}{1+x}\right)^2+\ldots\right]$$

11. Prove that

$$\left(\frac{1+x}{1-x}\right)^n=1+n.\,\frac{2x}{1+x}+\frac{n(n-1)}{2\,!}\left(\frac{2x}{1+x}\right)^2+\ldots \text{(P. U. 1916)}$$

12. Expand $(1+x+x^2)^{-3}$ and find the co-efficient of x^6.

(*P. U.* 1919)

13. Find the greatest term in the expansion of $(1+x)^{\frac{13}{3}}$ where $x=\frac{3}{4}$.

Application of the Binomial Theorem.

10·7. The following examples illustrate some important applications, of the *Binomial Theorem.*

Ex. 5. Find the cube root of 1001 correct to 5 places of decimal.

Sol $(1001)^{\frac{1}{3}}=(1000+1)^{\frac{1}{3}}=1000^{\frac{1}{3}}(1+\frac{1}{1000})^{\frac{1}{3}}$

$$=10(1+\cdot 001)^{\frac{1}{3}}$$

$$=10\left\{1+\tfrac{1}{3}\times(\cdot 001)+\frac{\frac{1}{3}(\frac{1}{3}-1)}{1.2}.(\cdot 001)^2+\ldots\ldots\right\}$$

$$=10\{1+\tfrac{1}{3}\times\cdot 001-\tfrac{1}{9}\times\cdot 000001+\ldots\}$$

$$=10\{1+\cdot 0003333-\cdot 0000001+\ldots\}$$

$$=10\times 1\cdot 0003332=10\cdot 003332$$

$=10\cdot 00333$ correct to 5 places of decimal.

Ex. 6. Find the fifth root of 31 by the Binomial Theorem. (*P. U.* 1934)

Sol. $(31)^{\frac{1}{5}}=(32-1)^{\frac{1}{5}}=32^{\frac{1}{5}}(1-\frac{1}{32})^{\frac{1}{5}}$

$$=2\left(1-\frac{1}{2^5}\right)^{\frac{1}{5}}$$

$$=2\left\{1+\frac{1}{5}\left(-\frac{1}{2^5}\right)+\frac{\frac{1}{5}(\frac{1}{5}-1)}{1.2}\left(-\frac{1}{2^5}\right)^2+\frac{\frac{1}{5}(\frac{1}{5}-1)(\frac{1}{5}-2)}{1.2.3}\left(-\frac{1}{2^5}\right)^3+\ldots\ldots\right\}$$

$$=2\left\{1-\frac{1}{5}\cdot\frac{1}{2^5}-\frac{1}{5}\cdot\frac{4}{5}\cdot\frac{1}{2}\cdot\frac{1}{2^{10}}-\frac{1}{5}\cdot\frac{4}{5}\cdot\frac{9}{5}\cdot\frac{1}{6}\cdot\frac{1}{2^{15}}-\ldots\ldots\right\}$$

$$=2\left\{1-\frac{1}{10}\cdot\frac{1}{2^4}\cdot-\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{2^7}\ \frac{1}{10}\cdot\frac{1}{10}\cdot\frac{3}{10}\cdot\frac{1}{2^{11}}-\ldots\ldots\right\}$$

$$=2\left\{1-\frac{1}{10}(\cdot 0625)-\frac{1}{100}(\cdot 0078125)-\frac{3}{1000}\cdot(\cdot 00048828)\ldots\right\}$$

$$=2\{1-\cdot 00625-\cdot 000078125-\cdot 00000146484\ldots\}$$

$$=2\{1-\cdot 00625-\cdot 00007k-\cdot 000001\ldots\}$$

$$=2\{1-\cdot 006329\}=2\times(993671)$$

$=1\cdot 987342=1\cdot 98734$, correct up to four decimal places.

Ex. 7. If x be nearly equal to 1, show that

$$\frac{px^p-qx^q}{p-q}=x^{p+q} \text{ nearly}$$

Sol. Let $x=1+h$, so that h is very small.

$$\text{L.H.S.}=\frac{px^p-qx^q}{p-q}=\frac{p(1+h)^p-q(1+h)^q}{p-q}$$

$$=\frac{p(1+ph+\dots)-q(1+qh+\dots)}{p-q}$$

where dots denote terms containing squares and higher powers of h. Now h being small, we may neglect these terms. Therefore

$$\text{L.H.S.}=\frac{p(1+ph)\quad q(1+qh)}{p-q}$$

$$=\frac{(p-q)+(p^2-q^2)h}{p-q}=1+(p+q)h, \text{approximately.}$$

Likewise $\text{R.H.S.}=x^{h+q}=(1+h)^{p+q}=1+(p+q)h+\dots\dots\dots$ where dots denote terms containing h^2 and higher powers of h. Neglecting these terms the R.H.S. is approximately equal to $1+(p+q)h$.

Hence L.H.S.=R.H.S. approximately.

Ex. 8. Show that if n and N are nearly equal

$$\left(\frac{N}{n}\right)^{\frac{1}{2}}=\frac{N}{N+n}+\frac{n+N}{4n} \text{ nearly} \qquad (P.\ U.\ 1925)$$

Sol. Let $N=n+h$, where h is so small that its squares and higher powers can be neglected.

$$\therefore\ \text{L.H.S.}=\left(\frac{N}{n}\right)^{\frac{1}{2}}=\left(\frac{n+h}{n}\right)^{\frac{1}{2}}=\left(1+\frac{h}{n}\right)^{\frac{1}{2}}$$

$$=1+\frac{1}{2}\cdot\frac{h}{n}+\text{terms containing } h^2, h^3 \text{ etc.}$$

$$=1+\frac{1}{2}\cdot\frac{h}{n} \text{ nearly.}$$

Also R.H.S.$=\frac{N}{N+n}+\frac{n+N}{4n}=\frac{n+h}{2n+h}+\frac{2n+h}{4n}$

$$=(n+h).\frac{1}{2n}\left(1+\frac{h}{2n}\right)^{-1}+\frac{1}{2}+\frac{h}{4n}$$

$$=\frac{1}{2n}(n+h)\left(1-\frac{1}{2}\cdot\frac{h}{n}\right)+\frac{1}{2}+\frac{h}{4n}$$

(neglecting h^2, h^3 etc.)

$$=\frac{1}{2n}\left(n-\frac{1}{2}h+h\right)+\frac{1}{2}+\frac{h}{4n},$$

(again neglecting h^2,)

$$=\frac{1}{2}-\frac{h}{4n}+\frac{h}{2n}+\frac{1}{2}+\frac{h}{4n}$$

$$=1+\frac{h}{2n}.$$

$\therefore$ L.H.S.=R.H S. approximately.

Ex. 9. Find the co-efficient of x^n in $\frac{(1+x)^2}{(1-x)^2}$.

Sol. $\frac{(1+x)^2}{(1-x)^2}=(1+x)^2(1-x)^{-2}$

$$=(1+2x+x^2)(1+2x+3x^2+4x^3+\ldots+(n+1)x^n+\ldots).$$

Hence the co-efficient of x^n in $\frac{(1+x)^2}{(1-x)^2}$

$=1\times$co-efficient of x^n in $(1-x)^{-2}$
$+2\times$co-effieient of x^{n-1} in $(1-x)^{-2}$
$+1\times$co-efficient of x^{n-2} in $(1-x)^{-2}$
$=1\times(n+1)+2+(n+1)+(n-1)$
$=4n.$

Ex. 10. Find the co-efficient of x^n in

$$\frac{(1-x)^3}{(1+3x)^4}.$$

Sol. We have

$$\frac{(1-x)^3}{(1+3x^4)}=(1-x)^3(1+3x)^{-4}$$

$$=(1-3x+3x^2-x^3)(1+P_1x+P_2x^2+\ldots+P_nx^n+\ldots)$$

where P_n denotes the co-efficient of x^n in the expansion of $(1+3x)^{-4}$.

i. e., $$P_n=\frac{-4(-4-1)(-4-2)...(-4-n+1)}{1.2.3...n}.3^n$$

$$=(-1)^n\frac{(n+1)(n+2)(n+3)}{1.2.3}.3^n$$

$\therefore$ the required co-efficient

$$=P_n-3P_{n-1}+3P_{n-2}-P_{n-3}$$

$$=(-1)^n\frac{(n+1)(n+2)(n+3)}{1.2.3}.3^n$$

$$-3.(-1)^{n-1}\frac{n(n+1)(n+2)}{1.2.3}.3^{n-1}$$

$$+3.(-1)^{n-2}.\frac{(n-1)n(n+1)}{1.2.3}3^{n-2}$$

$$-(-1)^{n-3}.\frac{(n-2)(n-1)n}{1.2.3}3^{n-3}$$

$$=\frac{(-1)^n}{1.2.3}.3^{n-3}[3^3(n+1)(n+2)(n+3)+3^3n(n+1)(n+2)$$

$$+3^2.(n-1)n(n+1)+(n-2)(n-1)n]$$

$$=(-1)^n\frac{3^{n-4}}{2}[64n^3+240n^2+344n+162]$$

$$=(-1)^n3^{n-4}(32n^3+120n^2+172n+81).$$

Ex. 11. Find the sum of the first $r+1$ co-efficients in the expansion of $(1-x)^{-n}$.

Sol. Here it is required to evaluate

$$1+n+\frac{n(n+1)}{1.2}+\frac{n(n+1)(n+2)}{1.2.3}+.....$$

$$.....+\frac{n(n+1)(n+2)...(n+r-1)}{1.2.3...r}$$

We have $$(1-x)^{-n}=1+nx+\frac{n(n+1)}{1.2}x^2+\frac{n(n+1)(n+2)}{1.2.3}x^3+...$$

$$...+\frac{n(n+1)(n+2)...(n+r-1)}{1.2.3...r}x^r+...$$

Also $(1-x)^{-1}=1+x+x^2+x^3+...+x^r+...$

Therefore if we multiply the two together, we see that the required value is the co-efficient of x^r in $(1-x)^{-n}\times(1-x)^{-1}$ *i. e.,* $(1-x)^{-(n+1)}$.

Hence the required sum

$$=\frac{(n+1)(n+2)\ldots(\overline{n+1}+r-1)}{1.2\ldots r}$$

$$=\frac{(n+1)(n+2)\ldots(n+r)}{r\,!}.$$

Ex. 12. Identify the following series as a Binomial expansion and hence find out its sum.

$$1+\tfrac{15}{8}+\tfrac{15}{8}\cdot\tfrac{21}{16}+\tfrac{15}{8}\cdot\tfrac{21}{16}\cdot\tfrac{27}{24}+\ldots\text{to infinity.}$$

(*P. U.* 1919, 1929)

Sol.

The series $=1+\frac{5}{2}\cdot\frac{3}{4}+\dfrac{\frac{5}{2}\cdot\frac{7}{2}}{4.8}\cdot 3^2+\dfrac{\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{4.8.12}\,3^3+\ldots.$

$$=1+\tfrac{5}{2}\cdot\tfrac{3}{4}+\frac{\frac{5}{2}(\frac{5}{2}+1)}{1.2}\cdot(\tfrac{3}{4})^2+\frac{\frac{5}{2}(\frac{5}{2}+1)(\frac{5}{2}+2)}{1.2.3}\cdot(\tfrac{3}{4})^3+\ldots\ldots$$

$$=(1-\tfrac{3}{4})^{-\frac{5}{2}}=(\tfrac{1}{4})^{-\frac{5}{2}}=4^{\frac{5}{2}}=2^5=32.$$

Note. Here all the terms are positive and the successive factors in the different terms go on increasing. This indicates that if at all the given series is a Binomial expansion, it must be obtained from $(1-x)^{-n}$. Hence, as a first step we transform the factors in the numerators in such a way that they go on increasing by unity. As a second step, we change the denominators in such a way that they contain the product of first **2, 3, 4**,...natural numbers respectively.

Second Method :—

Let the given series be identical with $(1+x)^n$

$$=1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots\ldots$$

Hence, we have

$$nx=\tfrac{15}{8} \qquad \ldots\ldots(i)$$

$$\frac{n(n-1)}{1.2}x^2=\tfrac{15}{8}\cdot\tfrac{21}{16} \qquad \ldots\ldots(ii)$$

$$\frac{n(n-1)(n-2)}{1.2.3}x^3=\tfrac{15}{8}\cdot\tfrac{21}{16}\cdot\tfrac{27}{24} \qquad \ldots\ldots(iii)$$

If the given series is a Binomial expansion, the values of x and n, as obtained from (*i*) and (*ii*) must satisfy (*iii*).

From (*i*) and (*ii*) by division

$$\frac{n-1}{2}x=\frac{21}{16}, \quad i.e., \quad nx-x=\frac{21}{8}$$

$$\therefore \quad \tfrac{15}{8}-x=\tfrac{21}{8} \quad \text{or} \quad x=-\tfrac{6}{8}=-\tfrac{3}{4}$$

From (i) $n=\frac{15}{8}\cdot\frac{1}{x}=\frac{15}{8}\cdot(-\frac{4}{3})=-\frac{5}{2}$

These values of x and n *do* satisfy (iii).

Hence the given series is actually a Binomial expansion and its sum is equal to $(1-\frac{3}{4})^{-\frac{5}{2}}=32$ as before.

Exercises

1. Find to 4 places of decimals $\sqrt{99}$. (*P. U.* 1919)
2. Find to 4 places of decimals $\sqrt{101}$. (*P. U.* 1923)
3. Find to 4 places of decimals $(244)^{\frac{1}{5}}$. (*P. U.* 1927)
4. Find to 5 places of decimals $(\cdot 998)^{-\frac{1}{3}}$. (*P. U.* 1940)
5. Evaluate $(1010)^{\frac{1}{3}}$ to 4 places of decimals. (*P. U.* 1924)
6. Obtain the 5th root of 31 to 3 places of decimals. (*P. U.* 1934)
7. Evaluate $(1+\frac{3}{10})^{\frac{8}{5}}$ to 4 decimal places. (*P. U.* 1926)
8. Evaluate (i) $(\frac{9}{10})^{\frac{4}{5}}$ to 4 decimal places. (*P. U.* 1926, 27, 29)

   (ii) $(344)^{\frac{1}{5}}$ to 4 decimal places. (*P. U.* 1927)

   (iii) $(129)^{\frac{1}{7}}$ to 3 decimal places.
9. If x be so small that its squares and higher powers may be neglected, show that :—

   (i) $\dfrac{1+x}{1-x}=1+2x.$ (*P. U.* 1934)

   (ii) $\dfrac{(9+2x)^{\frac{1}{2}}\ (3+4x)}{\sqrt[5]{1-x}}=9+16\frac{4}{5}x.$ (*Dehli* 1936)

10. If x be so small that its cubes and higher powers may be neglected, prove that

$$\frac{(1-x)^{-\frac{5}{2}}+(16+8x)^{\frac{1}{2}}}{(1+x)^{-\frac{1}{2}}+(2+x)^{2}}=1+\tfrac{23}{40}x^2. \qquad (P.\ U.\ 1933)$$

11. If x is nearly equal to 1, prove that

$$mx^m-nx^n=(m-a)x^{m+n}$$

approximately. (*P. U.* 1925)

12. If $p-q$ is small as compared to p or q then

$$\left\{\frac{p}{q}\right\}^{\frac{1}{n}}=\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} \text{ nearly.} \qquad (P.\ U.\ 1926)$$

13. If c is so small that c^3 and higher powers of c may be neglected in comparison with l^3, prove that

$$\sqrt{\frac{l}{l+c}}+\sqrt{\frac{l}{l-c}} \text{ is very nearly equal to } 2+\frac{3c^2}{4l^2}.$$

14. If the difference between x and 1 is very small, prove that

$$\frac{ax^b-bx^a}{x^b-x^a}=\frac{1}{1-x} \text{ very nearly.}$$

15. Expand $(1+x+x^2)^{-3}$ and find the co-efficient of x^6. (*P. U.* 1919)

16. Show that the co-efficient of x^n in the expansion of $\frac{1}{1+x+x^2}$ is 1, 0 or -1 according as n is of the form $3m$, $3m-1$ or $3m+1$.

17. Show that the co-efficient of x^n in the expansion of $\left\{\frac{1-x}{1-3x}\right\}^2$ is $(4n+8)3^{n-2}$.

18. Prove that the co-efficient of x^n in the expansion of $\frac{3x^2-2}{x+x^2}$ is $(-1)^{n-1}$.

19. Prove that the co-efficient of x^6 in the expansion of

$$(1+3x+6x^2+10x^3+\ldots)^3$$

is 3003.

20. Show that the sum of the first r co-efficients in the expansion of $(1-x)^n$ is

$$(-1)^{r-1}\frac{(n-1)(n-2)(n-3)\ldots(n-r+1)}{(r-1)\,!}.$$

21. Show that

$$\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}=1+\frac{x}{a+x}+\frac{3}{2}\cdot\left(\frac{x}{a+x}\right)^2+\frac{5}{2}\left(\frac{x}{a+x}\right)^3+\ldots$$

22. Show that

$$\left(\frac{1+x}{1-x}\right)^n=1+n\cdot\frac{2x}{1+x}+\frac{n(n+1)}{2\,!}\cdot\frac{4x^2}{(1+x)^2}+\ldots$$

(*P. U.* 1922)

23. Show that x^n may be expressed as

$$1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{1.2}\left(1-\frac{1}{x}\right)^2+\ldots$$

24. Indentify the following series as binomial expansions and hence find their sum :—

(*i*) $1+\frac{15}{8}+\frac{15}{8}\cdot\frac{21}{16}+\frac{15}{8}\cdot\frac{21}{16}\cdot\frac{27}{24}+\ldots$to infinity.

(*P. U.* 1929)

(*ii*) $1-\frac{1}{4}+\frac{1.3}{2.4}\cdot\frac{1}{2^2}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{2^3}+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{1}{2^4}-\ldots$

(*P. U.* 1938)

(*iii*) $1+\frac{7}{18}+\frac{7.9}{18.36}+\frac{7.9.11}{18.36.54}+\ldots\ldots$up to infinity.

(*P. U.* 1930)

(*iv*) $1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\ldots\ldots$ (*P. U.* 1942)

25. Prove that

$$\sqrt{2}=1+\frac{1}{2^2}+\frac{1.3}{2\,!}\cdot\frac{1}{2^4}+\frac{1.3.5}{3\,!}\cdot\frac{1}{2^6}+\ldots$$ (*P. U.* 1941)

26. If $y=\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\ldots\ldots\ldots\ldots$prove that $y^2+2y-2=0$. (*P. U.* 1939)

27. Prove that $(2+\sqrt{5})^{\frac{1}{3}}+(2-\sqrt{5})^{\frac{1}{3}}=1$. (*P. U.* 1922)

CHAPTER XI

Partial Fractions

11·1. The reader is already familiar with the process of algebraic addition by means of which a number of given fractions when added together give rise to single fraction, which may be called the 'resultant fraction'. It is sometimes, necessary to decompose a given fraction into its 'partial fractions' which when added would result in the given fraction. Like all inverse processes, this is little more difficult than the direct process of addition

It is not our object here to enter into a detailed theory of partial fractions. The different methods of obtaining partial fractions are fully illustrated by means of examples.

11·2. First of all, if in the fraction, which is required to be decomposed into partial fractions, the degree of the variable in the numerator is the same or greater than the degree of the variable in the denominator, it is necessary to divide out until the fraction is reduced to one in which the numerator is of lower degree than the denominator.

Secondly, the denominator should be resolved into elementary real factors. Four cases arise according as the denominator consists of

(*i*) linear non-repeated factors
(*ii*) quadratic non-repeated factors
(*iii*) linear repeated factors
(*iv*) quadratic repeated factors.

For example, in the denominator of

$$\frac{x}{(x-2)(x-3)^2(x^2+4)(x^2+3)^3}$$

$x-2$ is a linear non-repeated factor, $x-3$ is a linear factor repeated twice, x^2+4 is a quadratic non-repeated factor and x^2+3 is a quadratic factor repeated three times. Note that in the denominator of $\frac{1}{(x-2)(x^2-4)}$, x^2-4 is not a quadratic

non-repeated factor, for it can be resolved into two linear factors $(x+2)(x-2)$, so that in the denominator of the above fraction $x-2$ is a linear factor repeated twice, whereas $x+2$ is a non-repeated linear factor.

The third step is to suppose suitable partial fractions corresponding to each factor in the denominator involving certain unknown quantities which can be determined.

11·3. Non-repeated linear factors in the denominator. *Corresponding to each non-repeated linear factors $x-a$ in the denominator we may assume a partial fraction of the form* $\dfrac{A}{x-a}$.

Ex. 1. Decompose into partial fractions

$$\frac{x^2}{(x-1)(x-2)(x-3)}.$$

Sol. Here the degree of the numerator is lower than that of the denominator.

Let $\dfrac{x^2}{(x-1)(x-2)(x-3)} \equiv \dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$, where A, B, C are constants to be determined.

$\therefore \; x^2 \equiv A(x-2)(x-3)+B(x-3)(x-1)+C(x-1)(x-2)$.

Equating co-efficients of x^2, x and the absolute term on both sides of this identity, we have

$$A+B+C=1$$
$$-5A-4B-3C=0$$
$$6A+3B-2C=0$$

Solving these simultaneously, we get

$$A=\tfrac{1}{2},\; B=-4,\; C=\tfrac{9}{2}.$$

$$\therefore \frac{x^2}{(x-1)(x-2)(x-3)} \equiv \frac{1}{2}\cdot\frac{1}{x-1}-\frac{4}{x-2}+\frac{9}{2}\cdot\frac{1}{x-3}.$$

Otherwise thus :—

In the identity

$$x^2 \equiv A(x-2)(x-3)+B(x-3)(x-1)+C(x-1)(x-2).$$

Put $x=1$ (which is obtained by equating to zero, the factor in the denominator corresponding to which the co-efficient A has been supposed). We get $A=\frac{1}{2}$.

Similarly putting $x=2$ and $x=3$, respectively, we get the values of B and C.

Ex. 2. Decompose into partial fractions

$$\frac{2x^3}{(x-1)(x-3)(x-5)}.$$

Sol. Here the degree of the denominator is the same as that of the denominator. Therefore we first divide the numerator by the denominator. The quotient is easily seen to be 2 and we may suppose that

$$\frac{2x^3}{(x-1)(x-3)(x-5)} \equiv 2+\frac{A}{x-1}+\frac{B}{x-3}+\frac{C}{x-5},$$

where A, B, C are constants to be determined.

$\therefore$ We have $2x^3 \equiv 2(x-1)(x-3)\,x-5)+A(x-3)(x-5)$
$+B(x-5)(x-1)+C(x-1)(x-3)$.

Putting $x=1$, 3, 5 respectively we get

$$A=\frac{1}{4},\ B=\frac{-27}{2},\ C=\frac{125}{4}$$

$$\therefore \frac{2x^3}{(x-1)(x-3)(x-5)} \equiv 2+\ \frac{1}{4}\cdot\frac{1}{x-1}-\frac{27}{2}\cdot\frac{1}{x-3}+\frac{125}{4}\cdot\frac{1}{x-5}.$$

Ex. 3. Decompose into partial fractions

$$\frac{x^4+3x^3+7}{(x-1)(x-2)(x-3)}.$$

Sol. Here again the numerator is not of a lower degree than the denominator. But instead of actually dividing out we can suppose the quotient to be $x+A$.

Let $\dfrac{x^4+3x^3+7}{(x-1)(x-2)(x-3)} \equiv x+A+\dfrac{B}{x-1}+\dfrac{C}{x-2}+\dfrac{D}{x-3}$.

$\therefore x^4+3x^3+7 \equiv (x+A)(x-1)(x-2)(x-3)+B(x-2)(x-3)$
$+C(x-3)x-1)+D(x-1)(x-2)$

Putting $x=1$, 2, 3 respectively, we get

$$B=\tfrac{11}{2},\ C=-47.\ D=\tfrac{169}{2}$$

Again, equating the absolute terms on both sides of the identity, we have

$$7=-6A+6B+3C\div 2D$$

i. e., $A=\dfrac{6B+3C+2D-7}{6}=\dfrac{\frac{11}{2}\times 6+3\times(-47)+2\times\frac{169}{2}-7}{6}$

$=\dfrac{33-141+169-7}{6}=\dfrac{202-148}{6}$

$=\dfrac{54}{6}=9$

$$\therefore \quad \frac{x^4+3x^3+7}{(x-1)(x-2)(x-3)}\equiv x+9+\frac{11}{2}\cdot\frac{1}{x-1}-\frac{47}{x-2}+\frac{169}{2}\cdot\frac{1}{x-3}$$

Exercises

Decompose into partial fractions :—

1. $\dfrac{x+1}{x^2+5x+6}$. 2. $\dfrac{x}{(x-a)(x-b)}$ 3. $\dfrac{x^2+1}{x(x^2-1)}$.

4. $\dfrac{x^3}{(x+1)(x-2)(x+3)}$ 5. $\dfrac{2x^3-3x^2-8x-26}{2x^2-5x-12}$.

6. $\dfrac{(x-a)(x-b)}{(x-c)(x-d)}$. 7. $\dfrac{px+q}{(x-a)(x+b)}$.

8. $\dfrac{2x^2}{(x-a)(x-b)}$. 9. $\dfrac{x^4}{(x-1)(x-2)(x-3)}$.

10. $\dfrac{x^2-10x+13}{(x-1)(x^2-5x+6)}$. 11. $\dfrac{x-2}{(x-1)(x^2-5x+6)}$.

12. $\dfrac{2x^3+x^2-1}{2x^3+3x^2+x}$. 13. $\dfrac{(a-b)(b-c)(c-a)}{(ax-1)(bx-1)(cx-1)}$.

14. $\dfrac{x-4}{(x+4)(x^2-3x+2)}$. (*P. U.* 1936)

15. $\dfrac{x^2+px+q}{(x-a)(x-b)(x-b)}$. 16. $\dfrac{x^2+2}{(x^2+3)(x^2+4)}$.

17. $\dfrac{(x^2+a^2)(x^2+b^2)}{(x^2+c^2)(x^2+d^2)}$ 18. $\dfrac{f(x)}{(x-a)(x-b)(x-c)}$, where $f(x)$ denotes (*i*) a quadratic expression in x and (*ii*) a cubic expression in x in which the co-efficient of x^3 in R.

11·4. Non-Repeated quadratic factors in the denominator.

Corresponding to each quadratic non-repeated factor ax^2+bx+c in the denominator a partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$ must be supposed.

Ex. 4. Decompose into partial fractions
$$\frac{x+1}{(x^2+1)(x^2+x+1)}.$$

Sol. Let $\dfrac{x+1}{(x^2+1)(x^2+x+1)} \equiv \dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+x+1}$, where A, B, C, D are constants to be determined.

$$\therefore\ x+1 \equiv Ax+B)(x^2+x+1)+(Cx+D)(x^2+1).$$

Hence equating the co-efficients of x^3, x^2, x and the absolute terms on both sides of this identity, we have

$$A+C=0 \qquad \ldots(i)$$
$$A+B+D=0 \qquad \ldots(ii)$$
$$A+B+C=1 \qquad \ldots(iii)$$
$$B+D=1 \qquad \ldots(iv)$$

From (*i*) and (*iii*) by subtraction B=1,
Similarly from (*ii*) and (*iv*) A= −1,
Putting A= −1 in (*i*), we have C=1,
Putting B=1 in (*iv*), we have D=0.
Therefore
$$\frac{x+1}{(x^2+1)(x^2+x+1)} \equiv \frac{-x+1}{x^2+1}+\frac{x}{x^2+x+1}.$$

Ex. 5. Resolve $\dfrac{x^2+3}{(x-1)(x^2+1)(x^2+2)}$ into partial fractions.

Sol. Let $\dfrac{x^2+3}{(x-1)(x^2+1)(x^2+2)} \equiv \dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{x^2+2}$

So that
$$x^2+3 \equiv A(x^2+1)(x^2+2)$$
$$+(Bx+C)(x-1)(x^2+2)+(Dx+E)(x-1)(x^2+1)$$

Equating the co-efficients of x^4, x^3, x^2, x and the absolute term on both sides of the identity, we have

$$A+B+D=0 \qquad \ldots(i)$$
$$-B+C-D+E=0 \qquad \ldots(ii)$$

$$3A+2B-C+D-E=1 \quad \ldots(iii)$$
$$-2B+2C-D+E=0 \quad \ldots(iv)$$
$$2A-2C-E=3 \quad \ldots(v)$$

From (*ii*) and (*iv*) by subtraction, we see that
$$B=C \text{ and } D=E$$
$\therefore$ (*v*) becomes $2A-2B-D=0$;
This together with (*i*) gives on addition $B=3A$.
$\therefore$ (*iii*) becomes $3A+2\times 3A-3A+D-D=1$

i. e., $6A=1$ or $A=\frac{1}{6}$
$\therefore$ $B=3A=\frac{1}{2}=C$
$\therefore$ From (*i*) $D=-(A+B)=-(\frac{1}{6}+\frac{1}{2})=-\frac{2}{3}=E$
Thus $A=\frac{1}{6}$, $B=\frac{1}{2}$, $C=\frac{1}{2}$, $D=-\frac{2}{3}$, $E=-\frac{2}{3}$

and $$\frac{x^2+3}{(x-1)(x^2+1)(x^2+2)}\equiv\frac{1}{6}\cdot\frac{1}{x-1}+\frac{1}{2}\cdot\frac{x+1}{x^2+1}-\frac{2}{3}\cdot\frac{x+1}{x^2+2}.$$

Exercises

Resolve into partial fractions :—

1. $\dfrac{42-19x}{(x^2+1)(x-4)}$

2. $\dfrac{2x^2-10x+3}{(x-3)(x^2+2x+3)}$.

3. $\dfrac{x-4}{(x-1)(x-2)(x^2+4)}$. (*P. U.* 1936)

4. $\dfrac{x^4+2x^2+5}{x^4-3x^2+2}$.

5. $\dfrac{2x^3}{x^4-1}$.

6. $\dfrac{x^2+15}{(x-1)(x^2+2x+5)}$.

7. $\dfrac{x^4}{x^3-1}$.

8. $\dfrac{1}{x^3+1}$.

9. $\dfrac{3x+7}{(x^2+1)(x+3)}$. (*P. U.* 1940)

10. $\dfrac{1}{x^4+1}$.

11·5. Repeated linear factors in the denominator.

Corresponding to a linear factor $x-a$ repeated r times in the denominator, we must suppose r partial fractions

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\ldots+\frac{A_r}{(x-a)^r},$$

where $A_1, A_2, A_3, \ldots, A_r$ are constants to be determined.

Ex. 6. Resolve $\dfrac{4x^3}{(x+1)^3(x^2-1)}$ into partial fractions.

(*P. U.* 1942 *Supp.*)

Sol. Let $\dfrac{4x^3}{(x+1)^2(x^2-1)}=\dfrac{4x^3}{(x+1)^3(x-1)}\equiv\dfrac{A}{x-1}+\dfrac{B}{x+1}$

$$+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3}$$

$\therefore\ 4x^2\equiv A(x+1)^3+B(x-1)(x+1)^2+C(x-1)(x+1)+D'(x-1)$

Putting $x=1$ and -1, respectively, we have

$$A=\tfrac{1}{2},\ D=2.$$

Also equating the co-efficients of x^3 and the absolute terms on both sides, we have

$$A+B=4, \text{ giving } B=\tfrac{7}{2}\,;$$

and $A-B-C-D=0$ giving $C=A-B-D=-5$.

Thus $A=\frac{1}{2}$, $B=\frac{7}{2}$, $C=-5$, $D=2$.

Hence $\dfrac{4x^3}{(x+2)^2(x^2-1)}\equiv\dfrac{1}{2}\cdot\dfrac{1}{x-1}+\dfrac{7}{2}\cdot\dfrac{1}{x+1}-\dfrac{5}{(x+1)^2}$

$$+\frac{2}{(x+1)^3}.$$

Alternative Method :—

Put $x+1=y$ so that

$$\frac{4x^3}{(x+1)^3(x-1)}\equiv\frac{4(y-1)^3}{y^3(y-2)}=\frac{4}{y^3}\cdot\frac{y^3-3y^2+3y-1}{y-2}.$$

Divide y^3-3y^2+3y-1 by $y-2$ after re-arranging the terms in both these expressions in ascending order of powers of x. The process of division should be carried out until y^3 is a factor of the remainder. This is shown below :—

$$
\begin{array}{l}
-2+y)\,-1+3y-3y^2+y^3\,(\tfrac{1}{2}-\tfrac{5}{4}y+\tfrac{5}{8}y^3 \\
\qquad\quad \underline{-1+\tfrac{1}{2}y} \\
\qquad\qquad \tfrac{5}{2}y-3y^2 \\
\qquad\qquad \underline{\tfrac{5}{2}y-\tfrac{5}{4}y^2} \\
\qquad\qquad\quad -\tfrac{7}{4}y^2+y^3 \\
\qquad\qquad\quad \underline{-\tfrac{7}{4}y^2+\tfrac{7}{8}y^3} \\
\qquad\qquad\qquad\qquad \tfrac{1}{8}y^3
\end{array}
$$

Thus $\dfrac{y^3-3y^2+3y-1}{y-2}\equiv\frac{1}{2}-\frac{3}{4}y+\frac{7}{8}y^2+\frac{1}{8}\cdot\dfrac{y^3}{y-2}.$

Hence

$$\frac{4x^3}{(x+1)^2(x^2-1)}\equiv\frac{4}{y^3}\cdot\frac{y^3-3y^2+3y-1}{y-2}$$

$$\equiv\frac{4}{y^3}\left(\tfrac{1}{2}-\tfrac{5}{4}y+\tfrac{7}{8}y^2+\tfrac{1}{8}\cdot\frac{y^3}{y-2}\right)$$

$$\equiv\frac{2}{y^3}-\frac{5}{y^2}+\frac{7}{2}\cdot\frac{1}{y}+\frac{1}{2}\cdot\frac{1}{y-2}$$

$$=\frac{2}{(x+1)^3}-\frac{5}{(x+1)^2}+\frac{7}{2}\cdot\frac{1}{x+1}+\frac{1}{2}\cdot\frac{1}{x-1}.$$

Exercises

Resolve into partial fractions :—

1. $\dfrac{1}{(x^2+1)(x+1)^2}\cdot$ (*P. U.* 1935)

2. $\dfrac{2x+1}{(x+2)(x-3)^2}.$ (*P. U.* 1938)

3. $\dfrac{x^2+x+1}{(x-1)^3(x+1)}\cdot$ (*P. U.* 1939)

4. $\dfrac{x+4}{(x-2)^3(x+1)}.$ (*P. U.* 1941)

5. $\dfrac{1}{x^4(x+1)}.$

6. $\dfrac{x^3}{(x-a)^2(x^2+a^2)}.$

7. $\dfrac{3(x+1)}{(x-1)^2(x+2)^2}.$

8. $\dfrac{3-2x^2}{(x^2-3x+2)^2}.$

9. $\dfrac{x^2+x+1}{(x-1)^3(x+1)}.$

10. $\dfrac{3x^3-8x^2+10}{(x-1)^4}.$

11·6. Quadratic Repeated factors in the denominator.

Corresponding to each quadratic factor ax^2+bx+c repeated r times in the denominator, we must suppose r partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c},\ \frac{A_2x+B_2}{(ax^2+bx+c)^2},\ \ldots\ldots\frac{A_rx+B_r}{(ax^2+bx+c)_r}$$

Ex. 7. Resolve into partial fractions $\frac{x^2}{(1+x)(1+x^2)^2}$.

Sol. Assume $\frac{x^2}{(1+x)(1+x^2)^2}\equiv\frac{A}{1+x}+\frac{Bx+C}{1+x^2}+\frac{Dx+E}{(1+x^2)^2}$

$\therefore\ x^2\equiv A(1+x^2)^2+(Bx+C)(1+x)(1+x^2)+(Dx+E)(1+x)$.

Putting $x=-1$ on both sides of the identity, we have $A=\frac{1}{4}$.

Equating the co-efficients of x^4 on both sides, we have

$$A+B=0,\ i.e.,\ B=-\tfrac{1}{4}.$$

Fquating the co-efficients of x^3 on both sides, we get

$$B+C=0 \text{ giving } C=\tfrac{1}{4}.$$

Again equating the co-efficients of x^2, we have

$$2A+B+C+D=1 \text{ giving } D=\tfrac{1}{2}.$$

Lastly equating the co-efficients of x, we have

$$B+C+D+E=0, \text{ which gives } E=-\tfrac{1}{2}.$$

$$\therefore\ \frac{x^2}{(1+x)(1+x^2)^2}\equiv\frac{1}{4}\cdot\frac{1}{1+x}-\frac{1}{4}\cdot\frac{x-1}{x^2+1}+\frac{1}{2}\cdot\frac{x-1}{(x^2+1)^2}.$$

Exercises

Decompose into partial fractions :—

1. $\frac{x^2}{(1-x)(1+x^2)^3}$.
2. $\frac{3x+2}{(1-x)(1+x^2)^2}$.
3. $\frac{7}{(x+2)(x^2+2)^2}$.
4. $\frac{x+2}{(x+1)^3(x^2+1)^2}$.
5. $\frac{x^2+x+2}{x^2(x^2+3)^3}$.

11·7. Applications of partial fractions.

The following examples illustrate the use of partial fractions :—

Ex. 8. Expand $\frac{1}{(1-x)(1-2x)(1-3x)}$ in ascending powers of x and show that the co-efficient of x^n is $\frac{1}{2}(3^{n+2}-2^{n+3}+1)$.

(*P. U.* 1942)

Sol. Let $\dfrac{1}{(1-x)(1-2x)(1-3x)}\equiv\dfrac{A}{1-x}+\dfrac{B}{1-2x}+\dfrac{C}{1-3x}.$

So that $1\equiv A(1-2x)(1-3x)+B(1-3x)(1-x)+C(1-x)(1-2x)$

Putting $x=1$, we get $A=\frac{1}{2}$;

" $x=\frac{1}{2}$ " " $B=-4$;

" $x=\frac{1}{3}$ " " $C=\frac{9}{2}$.

Hence
$$\frac{1}{(1-x)(1-2x)(1-3x)}\equiv\frac{1}{2}\cdot\frac{1}{1-x}-4.\frac{1}{1-2x}+\frac{9}{2}\cdot\frac{1}{1-3x}$$
$$\equiv\tfrac{1}{2}(1-x)^{-1}-4(1-2x)^{-1}+\tfrac{9}{2}(1-3x)^{-1}.$$
$$\equiv\tfrac{1}{2}(1+x+x^2+x^3+\ldots\ldots+x^n+\ldots\ldots)$$
$$-4(1+2x+2^2x^2+2^3x^3+\ldots+2^nx^n+\ldots)$$
$$+\tfrac{9}{2}(1+3x+3^2x^2+3^3x^3+\ldots+3^nx^n+\ldots)$$

Hence the required co-efficient of x^n
$$=\tfrac{1}{2}\times1-4\times2^n+\tfrac{9}{2}\times3^n=\tfrac{1}{2}(3^{n+3}-2^{n+3}+1).$$

Ex. 9. Sum to n terms the series
$$\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\ldots\ldots$$
and deduce the sum to infinity.

Sol. Here
$$T_n=\frac{1}{(n\text{th term of } 3, 5, 7,\ldots)\times(n\text{ th term of } 5, 7, 9,\ldots)}$$
$$=\frac{1}{(2n+1)(2n+3)}$$
$$=\frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right),$$ by the method of partial fractions.

Putting $n=1, 2, 3$ etc ,
$$T_1=\tfrac{1}{2}(\tfrac{1}{3}-\tfrac{1}{5})$$
$$T_2=\tfrac{1}{2}(\tfrac{1}{5}-\tfrac{1}{7})$$
$$T_3=\tfrac{1}{2}(\tfrac{1}{7}-\tfrac{1}{9})$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$T_n=\tfrac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)$$

$\therefore$ By addition $S_n=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2n+3}\right)$

Now as $n\to\infty$, $S_n\to\frac{1}{6}$; therefore the sum to infinity of the series$=\frac{1}{6}$.

Ex. 9. Sum to n terms the series

$$\frac{1}{(1+r)(1+r^2)}+\frac{r}{(1+r^2)(1+r^3)}+\frac{r^2}{(1+r^3)(1+r^4)}+\ldots$$

Sol. Here $T_n=\dfrac{r^{n-1}}{(1+r^n)(1+r^{n+1})}$

Also $\dfrac{1}{1+r^n}-\dfrac{1}{1+r^{n+1}}=\dfrac{r^{n+1}-r^n}{(1+r^n)(1+r^{n+1})}=(r^2-r)T_n$

i.e., $T_n=\dfrac{1}{r^3-r}\cdot\left(\dfrac{1}{1+r^n}-\dfrac{1}{1+r^{n+1}}\right)$.

Hence putting $n=1, 2, 3$ etc, we have

$$T_1=\frac{1}{r^2-r}\left(\frac{1}{1+r}-\frac{1}{1+r^2}\right)$$

$$T_2=\frac{1}{r^2-r}\left(\frac{1}{1+r^2}-\frac{1}{1+r^3}\right)$$

$$T_3=\frac{1}{r^2-r}\left(\frac{1}{1+r^3}-\frac{1}{1+r^4}\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$T_n=\frac{1}{r^3-r}\left(\frac{1}{1+r^n}-\frac{1}{1+r^{n+1}}\right).$$

Hence by addition

$$S_n=\frac{1}{r^2-r}\left(\frac{1}{1+r}-\frac{1}{1+r^{n+1}}\right)$$

$$=\frac{1}{r(r-1)}\cdot\frac{r(r^n-1)}{(1+r)(1+r^{n+1})}=\frac{r^n-1}{(r^2-1)(r^{n+1}+1)}.$$

Exercises

1. Show that the co-efficient of x^n in the expansion of $\dfrac{1}{(1-x)(1-cx)(1-c^2x)}$ is $\dfrac{(1-c^{n+1})(1-c^{n+2})}{(1-c)(1-c^2)}$.

2. Prove that the co-efficient of x^n in the expansion of $\dfrac{(p-q)^2}{(1-px)(1-qx)}$ is $(n+1)p^{n+2}-(n+2)p^{n+1}.q+q^{n+2}$.

3. Prove that the co-efficient of x^n in the expansion of $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$ is

$$\frac{a^{n+2}}{(a-b)(a-c)}+\frac{b^{n+2}}{(b-c)(b-a)}+\frac{c^{n+2}}{(c-a)(c-b)}.$$

4. Expand $\dfrac{1}{x^2-5x+6}$ in the forms

(i) $p_0+p_1x+p_2x^2+\ldots$

and (ii) $q_0+\dfrac{q_1}{x}+\dfrac{q_2}{x^2}+\ldots$

Hence show that $q_{n+1}=6^n p_{n-1}$.

5. If $\dfrac{3x-2}{x^2-5x+6}=p_0+p_1x+p_2x^2+\ldots+p_nx^n+\ldots\ldots$, prove that $p_0+p_1+p_2+\ldots+p_n=4\left(1-\dfrac{1}{2^{n+1}}\right)-\dfrac{7}{2}\left(1-\dfrac{1}{3^{n+1}}\right)$.

6. Sum up to n terms and to infinity the series

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots$$ (P. U. 1934)

7. Sum up to n terms and to infinity the series

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots$$

8. If $r<1$, sum the following series to n terms and deduce the sum to infinity :—

$$\frac{1}{(1+x)(1+rx)}+\frac{r}{(1+rx)(1+r\cdot x)}+\frac{r^2}{(1+r^2x)(1+r^3x)}+\ldots$$

9. Sum to $2n$ terms the series.

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\ldots$$

Also find the sum to infinity.

10. Sum to n terms the series

$$\frac{1}{(1+x)(1+2x)}+\frac{2}{(1+2x)(1+4x)}+\frac{4}{(1+4x)(1+8x)}+\ldots$$

PUNJAB UNIVERSITY PAPERS

1939

1. Discuss the nature of the roots of the quadratic equation $ax^2+bx+c=0$, and show that the imaginary roots occur in pairs.

Find the equation whose roots are the arithmetic and harmonic means between the roots of the equation

$$x^2+2px+q=0.$$

Solve :

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}-\frac{7}{\sqrt{xy}}=1,$$

$$\sqrt{xy}(x+y)=78.$$

2. Sum the series

(i) $a+(a+b)r+(a+2b)r^2+(a+3b)r^3+\ldots\ldots$to infinity, r being a proper fraction.

(ii) ${}^2C_2+{}^3C_2+{}^4C_2+\ldots\ldots$to n terms.

A geometrical and a harmonical progression have the same pth, qth and rth terms a, b, c respectively. Show that

$$a(b-c)\log a+b(c-a)\log b+c(a-b)\log c=0.$$

3. Find the number of permutations of n dissimilar things taken r at a time In how many of them will three particular things always occur ?

In how many ways can 16 departments be allotted to 4 ministers so that no minister shall have less than 3 departments under him ?

4. Find the $(n+1)th$ term from the end in the expansion of

$$\left(x-\frac{1}{x}\right)^{3n}$$

Evaluate $\sqrt[3]{2}$ to two places of decimal.

If $y=\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\ldots$to infinity, prove that

$$y^2+2y-2=0.$$

Or

Split up into partial fractions the following :—

$$\frac{x^2+x+1}{(x-1)^3(x+8)}.$$

1940

1. Find the condition that one root of the equation $x^2+px+q=0$ may be (1) square of the other, (2) double of the other.

Solve :—

$$x+y=5$$

$$\frac{2}{x}+\frac{3}{y}=2.$$

2. The sum of four numbers in G. P. is 60, and the arithmetic means of the first and the last is 18. Find the numbers.

Sum the series

$$1.n+2(n-1)+3.(n-2)+\ldots\ldots+n.1.$$

3. Prove

(1) ${}^nC_r={}^nC_{n-r}$.

(2) ${}^nC_0+{}^nC_1+{}^nC_2+{}^nC_3+\ldots{}^nC_n=2^n$.

The Premiers of 11 provinces of India meet to discuss the problems of the minorities. In how many ways can they seat themselves at a round table if the Punjab and Bengal Premiers choose to sit together ?

4. State the Binomial Theorem for positive integral index, and find the term independent of x in the expansion of

$$\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9.$$

If $x<1$, find the co-efficient of x^n in the expansion of

$$(1+x+x^2+x^3+\ldots\ldots\infty)^2.$$

Or,

Split up into partial fractions the expression

$$\frac{3x+7}{(x^2+1)(x+3)}.$$

1941

1. (*a*) Discuss the nature of the roots of the quadratic
$$ax^2+bx+c=0.$$
(*b*) Solve the equations :—

(1) $x^2+y^2=13,\ xy=6$;

(2) $\sqrt{x^2-3x+36}-\sqrt{x^2-3x+9}=3.$

2. (*a*) Find the sum of the squares of the first n natural numbers.

Exaluate $\cdot 9\dot{7}$

(*b*) Sum the series

$1+3-5+7+9-11+13+15-17+\ldots\ldots$to $3n$ terms.

Or,

If S_1, S_2, S_3 be the sums of n, $2n$, $3n$ terms respectively of an A. P., show that $S_3=3(S_2-S_1)$.

3. (*a*) Find the number of permutations of n dissimilar things taken r at a time.

(*b*) Find how many arrangements can be made with the letters of the word mathematics', and in how many of these vowels occur together ?

Or,

Resolve into partial fractions
$$\frac{x+4}{(x-2)(x+1)}.$$

4. Find the term independent of x in the expansion of
$$\left(x^2-\frac{2}{x}\right)^9.$$

Find the value of $(\cdot 998)^{-\frac{1}{3}}$ correct to five decimal places.

Show that $\sqrt{2}=1+\frac{1}{2^2}+\frac{1.3}{2\,!\,2^4}+\frac{1.3\,5}{3\,!\,2^6}+\ldots\ldots$to infinity.

1942

1. (*a*) Prove that a quadratic equation cannot have more than two roots.

(*b*) Is it possible that a quadratic equation with real co-efficients may have one real and one imaginary root ? Give a proof of your answer.

(*c*) Find an equation whose roots are double the roots of $x^2-px+q=0$.

(*d*) Solve the equations

$$x+y=a+b,\ \frac{a}{x}+\frac{b}{y}=2.$$

2. (*a*) Find the sum of n terms in A. P.

(*b*) If a, b, c are in H. P., show that

$$\frac{a}{b+c},\ \frac{b}{c+a},\ \frac{c}{a+b} \text{ are also in H. P.}$$

(*c*) If the mth term of a series in H. P be n and the nth term be equal to m, prove that the $(m+n)$th term is equal to $\frac{mn}{m+n}$.

3. (*a*) Find the sum of the cubes of the first n natural numbers, and show that it is equal to the square of the sum of these numbers.

(*b*) Find the nth term of any one of the following series :—

$$4+11+22+37+66+\ldots\ldots$$

$$2\tfrac{1}{3}+1\tfrac{7}{13}+1\tfrac{1}{9}+\tfrac{20}{23}+\ldots\ldots$$

(*c*) Find the sum of the series

$$1^2+(1^2-2^2)+(1^2+2^2+3^2)+\ldots\ldots \text{to } n \text{ terms.}$$

4. (*a*) Find the number of ways in which n different beads can be arranged to form a necklace.

(*b*) A gentleman invites a party of 13 guests to a dinner and places 8 of them at one table and the remaining 5 at another, the tables being round. Find the number of ways in which he can arrange the guests.

(*c*) How many words can be formed out of the letters of the word 'article' ; so that the vowels may occupy the even place ?

5. (*a*) Prove the Binomial Theorem of the positive integral exponent.

(*b*) In the expansion of $(1+x)^n$, where n is a positive integer, prove that the co-efficients of terms equidistant from the beginning and end are equal.

(*c*) Express $\dfrac{1}{(1-x)(1-2x)(1-3x)}$

in the form of partial fractions, and hence show that the co-efficient of x^n in the given fraction is $\frac{1}{2}(3^{n+2}-2^{n+3}+1)$.

1943

1. (*a*) Find from the first principles the formula for the roots of the equation $ax^2+2bx+c=0$.

(*b*) Find the nature of the roots of the following equations :—

(1) $2x^2-7x+3=0$ (2) $x^2-5x-2=0$.

(*c*) For what value of m will the equation

$$(m+1)x^2+2(m+3)x+(2m+3)=0$$

have equal roots ?

(*d*) Form a quadratic equation with rational co-efficients, one of whose roots shall be $\dfrac{1}{2+\sqrt{5}}$.

2. (*a*) Find the formula for the sum of p terms of a G.P., whose first term is x and whose common ratio is t.

(*b*) How many terms of the series $\frac{2}{9}-\frac{1}{3}+\frac{1}{2}+\ldots\ldots$ must be taken to amount to $\frac{55}{72}$?

(*c*) Find five numbers in A. P. whose sum is 25 and the sum of whose squares is 165.

3. (*a*) Find the formula for the sum Σn^2.

(*b*) Sum to infinity the series

$$1^2+2^2x+3^2x^2+4^2x^3+\ldots\ldots$$

x being less than 1 (numerically).

(*c*) If a, b, c be in A. P. ; p, q, r in H. P. ; ap, bq, cr in G. P., then show that

$$\frac{p}{r}+\frac{r}{p}=\frac{a}{c}+\frac{c}{a}.$$

4. (*a*) Find the number of permutations of n dissimilar things taken r at a time. In how many of these will four given things be excluded ?

(*b*) How many different numbers of six digits can be formed with the digits 3, 1, 7, 0, 9, 5 ? How many of these will have 0 in the ten's place ?

5. (*a*) Give a precise statement of the Binomial Theorem when the index n is not a positive integer.

Find in its simplest form the $(r+1)$th term in the expansion $(1-x)^{-n}$,

(*b*) Resolve into partial fractions

$$\frac{x+a}{x^2(x-a)(x+a^2)}.$$

1944

1. (*a*) Without actually solving it, prove that the equation $ax^2+bx+c=0$ cannot have more than two roots.

(*b*) Solve the equations

$$x+y=5$$
$$x^2+2y^2=17.$$

(*c*) Find the value of m for which the equation

$$(m+1)x^2+2(m+3)x+(m+8)=0$$

has equal roots.

2. (*a*) Find the sum of n-terms in A. P., the first term being x and the common difference being y.

(*b*) Find the 14 arithmetic means which can be inserted between 5 and 8, and show that their sum is 14 times the arithmetic mean between 5 and 8.

(*c*) Find three numbers in G. P. whose sum is 19 and whose product is 216.

3. (*a*) Find the sum of the cubes of the first n natural numbers.

(*b*) Sum the series

$$1^3+3^3+5^3+\ldots\text{to } n \text{ terms.}$$

(*c*) If $b+c$, $c+a$, $a+b$ are in H. P., show that a^2, b^2, c^2 are in A. P.

4. (*a*) If ${}^nP_r$ denotes the number of permutations of n things taken r at a time, prove from first principles that

$${}^nP_r = n \times {}^{n-1}P_{r-1}.$$

Hence find the value of ${}^nP_r$ as a product.

(*b*) How many permutations can be made out of the letters of the word *triangle*? How many of these will begin with t and end with e?

(*c*) Use the Binomial Theorem to expand $(1-x+x^2)^4$ in ascending powers of x.

5. (*a*) Find the cube root of 126 to 4 places of decimals.

(*b*) Resolve into partial fractions—

(*i*) $\dfrac{x^2+1}{x^3+1}$.

(*ii*) $\dfrac{1}{(x+1)^2(x^4+1)}$.

1945

1. (*a*) Solve the equation

$$ax^2+bx+c=0.$$

Discuss the nature of the roots.

Find the condition that one root may be square of the other.

(*b*) Solve the equations

$$\frac{1}{x^4}+\frac{1}{y^4}=\frac{5}{16}$$

$$\frac{1}{x}+\frac{1}{y}=\frac{3}{4}.$$

2. (*a*) Find the sum of n terms of a G. P. series whose first term is a and the common ratio is r.

Deduce the sum of an infinite G. P. series, the absolute value of r being less than one.

(*b*) Sum the series

(*i*) $1+\dfrac{2}{5}+\dfrac{3}{5^2}+\dfrac{4}{5^3}+\ldots$ to n terms.

(*ii*) $6^2+7^2+8^2+\ldots\ldots+20^2$.

3. (*a*) Prove that with the usual notation

$$^{n}C_r=\frac{n}{r\,!\,(n-r)\,!}.$$

Deduce that $\dfrac{n(n+1)(n+2)\ldots(n+r-1)}{r\,!}$ is an integer.

(*b*) How many numbers of six digits can be formed from the digits 4, 5, 6, 7, 8, 9, no digit being repeated ?

How many of them are not divisible by 5 ?

4. (*a*) Prove Binomial Theorem for a positive integral index.

Show in the expansion of $(1+x)^n$ the sum of the co-efficients of odd terms is equal to the sum of the co-efficients of even terms, and each one of the sums is equal to 2^{n-1}.

(*b*) If x be so small that its square and higher powers are neglected, show that

$$\frac{(9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}}{4+5x}=\frac{1}{4}-\frac{17}{384}x.$$

5. Resolve into partial fractions *any two* of the following :—

(*i*) $\dfrac{9x+7}{(x+3)(x^2+1)}$

(*ii*) $\dfrac{x^2}{(x-1)^3(x+2)}$

(*iii*) $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$.

ANSWERS

Page 5, 6.

1. $6, -10.$ 2. $a, b.$ 3. $\frac{5}{2}, -\frac{8}{3}.$
4. $-1, -4.$ 5. $4, -1.$
6. $\frac{1}{3}[(a+b+c) \pm \sqrt{a^2+b^2+c^2-bc-ca-ab}.]$
7. $\frac{3}{4}a, \frac{1}{2}a.$ 8 $2, 3.$ 9. $\sqrt{2}, 2\sqrt{2}.$
10. $1, 1+\sqrt{2}.$ 11. $2, \frac{58}{13}.$ 12. $\frac{11 \pm \sqrt{13}}{6}.$
13. $\frac{7}{8}.$ 14. $\frac{1}{3}(-6 \pm \sqrt{3}).$
15. $0, -\frac{a+b}{2}$ 16. $\frac{a+b}{ab}, \frac{a}{a+b}.$
17. $13.$ 18. $0, -\frac{5}{2}.$ 19. $0, \pm\frac{1}{2}\sqrt{a^2+b^2}.$

Page 9, 10.

1. $0.$ 2. $\pm\sqrt{\frac{2}{3}}.$ 3. $0, \pm\frac{\sqrt{3}}{2}.$
4. $\pm\frac{\sqrt{3}}{2}.$ 5. $\frac{b}{a}, \frac{2b}{a}\left[1 \pm \sqrt{7}\right].$ 6. $0.$
7. $-2, -3$ both extraneous.
8. $2, \frac{9+4\sqrt{3}}{3},$ the last two are extraneous.
9. $-1, \frac{89}{4}$, the last one being extraneous.
10. $\frac{157}{88}.$ 11. $6, 3$, the last one being extraneous.
12. $1.$ 13. $\frac{1}{3}[-(a+b+c) \pm 2\sqrt{a^2+b^2+c^2-bc-ca-ab})].$
15. $0, -2.$ 16. $1, -4$, where the last one is extraneous.
17. $2a, -3a$, where the last root is extraneous.
18. $0, 3.$ 19. $0, \frac{8}{5}$, the last one being extraneous.
20. $\frac{1}{8}(-7 \pm \sqrt{17}).$ 21. $0, -\frac{b}{a}.$

Page 14.

1. $\frac{10}{13}, \frac{14}{13}.$ 2. $1. 2, \frac{1}{2}(3 \pm \sqrt{17}).$ 3. $\pm 1, \pm\sqrt{3}.$
4. $2^n, 2^{-n}.$ 5. $16, (\frac{11}{8})^4.$ 6. $3.$

7. $1, -5, 2(-1\pm\sqrt{2})$.
8. $2, 6, \frac{1}{2}(-4\pm\sqrt{-59})$.
9. $a, -9a, a(-4\pm\sqrt{-15})$
10. $1, -2, -\frac{1}{2}, \frac{1}{6}(7\pm\sqrt{13})$.
11. $\pm 1, 1\pm\sqrt{2}$.
12. $2, \frac{1}{2}, -3, -\frac{1}{3}$.
13. $1, 1, -2, -\frac{1}{2}$.
14. $-1, -2, -\frac{1}{2}, \frac{1}{2}(3\pm\sqrt{5})$.

Page 15, 16.

1. $1, \dfrac{a-b}{b-c}$
2. $0, \frac{1}{2}[c-a-b\pm\sqrt{c^2-(a-b)^2}]$.
3. $a-2b, 2a-b$.
4. $a+b, -a-2b$.
5. $\dfrac{\sqrt[3]{2}+1}{\sqrt[3]{2}-1}$.
6. $1, 1, \frac{1}{2}(-3\pm\sqrt{5})$.
7. $\dfrac{2}{3}, \frac{1}{3}(-1\pm\sqrt{10})$.
8. $\dfrac{144}{7}, -\dfrac{256}{7}$.
9. $\pm\frac{1}{2}$.
10. $9, -7, 1\pm 2\sqrt{-6}$
11. $2\pm\sqrt{3}, \dfrac{-1\pm\sqrt{-3}}{2}$.
12. $\pm\sqrt{3}, \dfrac{2\pm\sqrt{-5}}{3}$.
13. $3, 3, 3, 3$.
14. 14^4.
15. $4, -\frac{7}{2}, \frac{1}{4}(1\pm\sqrt{65})$.
16. $2, -\frac{1}{2}, \frac{1}{4}(3\pm\sqrt{505})$.
17. $-\frac{1}{2}, 4$.
18. $1, \frac{1209}{639}$.
19. $0, a+b, \dfrac{a^2+b^2}{a+b}$.
20. $1, 2, -\frac{2}{3}$.
21. $\pm\sqrt{1-\dfrac{(a-2)^3}{27a}}$.
22. $\dfrac{a+b}{b}, \dfrac{a+b}{a}$.
23. length 5, breadth 3.
24. 10, 11.
25. -2
26. $1, -5$.
27. 150.

Page 19.

2. (*i*) $A=\frac{5}{2}, B=-15, C=\frac{29}{3}$.
   (*ii*) $A=\frac{[illegible]}{5}, B=\frac{3}{4}, C=\frac{31}{20}$.
   (*iii*) $A=1, B=\dfrac{a}{2}, C=\dfrac{-a}{4}$.

Page 24, 25.

1. (*i*) $x^2-5x+6=0$; (*ii*) $x^2-x-12=0$;
   (*iii*) $5x^2-26x+5=0$; (*iv*) $x^2-(\sqrt{3}+\sqrt{2})x+\sqrt{6}=0$;
   (*v*) $x^2-4x+1=0$; (*vi*) $4x^2-6x+1=0$;
   (*vii*) $nx^2-2mx+m=0$.
2. (*i*) $p=0$; (*ii*) $y=1$.

3. (*i*) $-\frac{b}{\sqrt{ac}}$, (*ii*) $\frac{b^2-4ac}{a^2}$ (*iii*) $\frac{b^2-3ac}{a^4}$,

(*iv*) $\frac{b^4-4ab^2c+2a^2c^2}{a^4(b^2-4ac)}$, (*v*) $p^3+\frac{p(b^2-2ac)+c^2}{a^2}$,

(*vi*) $p^2+\frac{bp\sqrt{b^2-4ac}-c^2}{a^2}$, (*vii*) $\frac{b^2}{a^2c^2}(b^2-4ac)$.

(*viii*) $\frac{(b^2-2ac)^2-2a^2c^2}{a^2c^2}$ (*ix*) $\frac{b^4-3ab^2c+a^2c^2}{ab(2ac-b^2)}$,

(*x*) $\frac{b^2-2ac}{c\,a^2}$.

4. 27, -125.

5. (*i*) $acx^2+b(c+a)x+b^2=0$.

(*ii*) $a^2c^2x^2-(b^2-2ac)(c^2+a^2)x+(b^2-2ac)^2=0$.

(*iii*) $(a-b+c)x^2+2(c-a)x+(a+b+c)=0$.

(*iv*) $a^4x^2-2a^2(b^2-2ac)x+b^2(b^2-4ac)=0$.

(*v*) $a^3c^3x^2-(a^3+c^3)(3abc-b^3)x+(3abc-b^3)^2=0$.

(*vi*) $a(a-b+c)x^2+(ah+2ac-b^2)x+ac=0$.

(*vii*) $acx^2+b(c+a)x+(c-a)^2+b^2=0$.

7. $2(q+q')-pp'$. 8. $\frac{1}{4q-p^2}$.

9. $x^2-(\alpha+\beta+\alpha\beta)x+\alpha\beta(\alpha+\beta)=0$.

Page 30, 31.

1. real, unequal and irrational. 2. imaginary.

3. roots are always real. They are rational, if either $c=0$ or $a=b$. They will be equal if $c=0$ and $a=b$.

1. real, distinct and rational. 5. $k=0$ or 3.

8. The roots are always imaginary. They will be equal also, if $\alpha=\beta$ or $\alpha=0$ or $\beta=0$.

13. $m=\frac{a}{c}$. 14. $2, -\frac{10}{9}$.

Page 34

1. (*i*) $3+29i$. (*ii*) $-\sqrt{6}-\sqrt{14}-\sqrt{15}-\sqrt{35}$.

2. (*i*) $\frac{23+14i}{21}$, (*ii*) $-\frac{58}{65}$.

3. (*i*) $\pm(1+i)$, (*ii*) $\pm(1-i)$, (*iii*) $\pm(3+4i)$, (*iv*) $\pm(3-4i)$

4. $-1, -w, -w^2$, 6. *i*) 2, (*ii*) $8-27i$.

9. (*i*) $2, 2w, 2w^2$, (*ii*) $3, 3w, 3w^2$, (*iii*) a, w, aw^2.

Page 41.

1. $x=3, y=2$; $x=\frac{3}{4}, y=-\frac{29}{8}$.
2. $x=2, y=1$; $x=\frac{5}{2}, y=\frac{3}{5}$.
3. $x=1, y=2$; $x=-2, y=-2$.
4. $x=0, y=0$; $x=2a, y=2b$.
5. $x=5, y=2$; $x=\frac{18}{3}, y=\frac{8}{9}$.
6. $x=5, y=12$; $x=169, y=\frac{169}{24}$.
7. $x=4, y=9$; $x=9, y=4$;
   $x=-3+4\sqrt{-10}, y=-3-4\sqrt{-10}$.
   $x=-3-4\sqrt{-10}, y=-3+4\sqrt{-10}$.
8. $x=2, y=8$; $x=8, y=2$.
9. $x=\frac{1}{3}, y=\frac{1}{2}$; $x=\frac{5}{2}, y=-\frac{5}{3}$.
10. $x=1, y=3, x=3, y=1$, $x=\dfrac{25+\sqrt{605}}{2}$, $y=\dfrac{25-\sqrt{-605}}{2}$
11. $x=3, y=6$; $x=6, y=3$, $x=\dfrac{25-\sqrt{-605}}{2}$, $y=\dfrac{25+\sqrt{605}}{2}$
12. $x=9, y=\pm 3$; $x=-9, y=\pm 3$

Page 43.

1. $x=1=y$; $x=-1=y$; $x=\pm 2\sqrt{\frac{3}{25}}$, $y=\pm 4\sqrt{\frac{3}{35}}$.
2. $x=\pm\sqrt{\frac{3}{2}}, y=\mp\sqrt{\frac{1}{2}}$; $x=\pm 2, y=\mp 1$.
3. $x=\pm 2, y=\pm 2$; $x=\pm 2\sqrt{\frac{2}{3}}, y=\pm 4\sqrt{\frac{2}{3}}$.
4. $x=\pm 3, y=\pm 2$; $x=\pm\dfrac{6}{\sqrt{7}}, y=\pm\dfrac{1}{\sqrt{7}}$.
5. $x=3, y=2$; $x=-3, y=-2$
6. $x=\pm 3, y=\pm 4$; $x=\pm 8\sqrt{\frac{37}{127}}, y=\pm 6\sqrt{\frac{37}{127}}$.
7. $x=\dfrac{b\pm\sqrt{2a^2-b^2}}{2}$, $y=\dfrac{b\mp\sqrt{2a^2-b^2}}{2}$.
8. $x=a\pm\sqrt{a^2-k^2}$, $y=a\mp\sqrt{a^2-k^2}$.

Page 44.

1. $x=1, y=5$; $x=5, y=1$.
2. $x=\dfrac{-5\sqrt{226}}{2}$, $y=\dfrac{-7+\sqrt{226}}{2}$,
   $x=\dfrac{-5-\sqrt{226}}{2}$, $y=\dfrac{-7-\sqrt{226}}{2}$
3. $x=\pm 4, \pm 5$; $y=\pm 5, \pm 4$.

4. $x=3, y=9$; $x=9, y=3$.
5. $x=a, y=0$; $x=0, y=b$.
6. $x=b, y=a$; $x=\dfrac{a^2}{b}, y=\dfrac{b^2}{a}$.
7. $x=1, y=\frac{1}{5}$; $x=\frac{1}{5}, y=1$.
8. $x=9, y=4$; $x=4, y=9$.
9. $x=1, y=3$; $x=3, y=1$.
10. $x=1, y=a$; $x=a, y=1$.

Page 45, 46.

1. $x=1, y=2, z=4$; $x=-1, y=-2, z=-4$.
2. $x=1, y=2, z=1$.
3. $x=-11\left(\dfrac{67}{706}\right)^{\frac{1}{3}}, \; y=14\left(\dfrac{67}{706}\right)^{\frac{1}{3}}, \; z=-\left(\dfrac{67}{706}\right)^{\frac{1}{3}}$
4. $x=2, y=3, z=4$; $x=\frac{34}{5}, y=-\frac{13}{5}, z=0$.
5. $x=\pm 2, y=\pm 3, z=\pm 4$.
6. $x=\pm 1, y=\pm 2, z=\pm 3$.
7. $x=\pm 1, y=\pm 2, z=\pm 3$.
8. $x=1, y=1, z=3$; $x=-3, y=-3, z=-5$.
9. $x=\frac{58}{9}, y=\frac{85}{18}, z=\frac{20}{10}$.
10. $x=6, y=9, z=1$.
11. $x=2, y=3, z=5$; $x=-5, y=\frac{85}{12}, z=\frac{95}{12}$.
12. $x=\pm\dfrac{b+c-a}{2}, \; y=\pm\dfrac{c+a-b}{2}, \; z=\pm\dfrac{a+b-c}{2}$.

Page 46 to 49.

I

2. $a^2x^2+(2ac-b^2)x+c^2=0$.
3. (*i*) 4, $\dfrac{10}{11}$; (*ii*) 1, $\dfrac{43 \pm \sqrt{-135}}{16}$, (*iii*) 6.
4. (*i*) $x=1, y=\frac{1}{2}$; $x=-1, y=-2$;
   (*ii*) $x=2, y=8$; $x=8, y=2$;
   (*iii*) $x=1, y=2$; $x=2, y=1$;

II

1. $p^2x^2-(2pr+q^2)x+r^2=0$.

4. (*i*) $2, \frac{-74}{35}$; (*ii*) $2, 4, 3 \pm 6\sqrt{2}$; (*iii*) $\pm \frac{62}{33}$.

5. (*i*) $x=2, y=3$: $x=3, y=2$;
(*ii*) $x=3, y=4$; $x=\frac{25}{8}, y=\frac{25}{8}$;
(*iii*) $x=\frac{30 \pm 12\sqrt{5}}{5}, \; y=\frac{30 \mp 12\sqrt{5}}{5}$.

III

3. 12. 4. (*i*) $-1, \; \frac{1+\sqrt{5} \pm \sqrt{2\sqrt{5}-10}}{4}$,

$\frac{1-\sqrt{5} \pm \sqrt{-2\sqrt{5}-10}}{4}$

(*ii*) $\frac{1}{5}, -\frac{1}{2}$; (*iii*) $2, 3, -\frac{1}{2}, -\frac{1}{3}$.

5. (*i*) $x=-6, x=\pm 3$; $x==9, y \pm 3$.
(*ii*) $x=7, y=11$; $x=-\frac{89}{5}, y=-\frac{56}{5}$;
(*iii*) $x=2, y=3$.

IV

3. $a(a-b+b)x^2+(ab+2ac-b^2)x+ac=0$.

4. (*i*) $0, \pm\frac{\sqrt{3}}{2}$; (*ii*) $0. \; a+b$,

5. (*i*) $x=\frac{8}{3}, y=\frac{56}{27}$; $x=\frac{63}{28}, y=\frac{9}{4}$.
(*ii*) $x=\pm 2, y=\pm 1$; $x=\pm\frac{3}{2}, y=\pm\frac{1}{2}$.
(*iii*) $x=\frac{1}{2}.\sqrt{26}\,(\sqrt{3} \pm 3i), \; y=\frac{1}{2}\sqrt{26}(\sqrt{3} \mp 3i)$.

V

4. (*i*) $\frac{ab(c+d)-cd(a+b)}{ab-cd}$, (*ii*) $\pm\sqrt{3}, \frac{2 \pm \sqrt{-5}}{3}$
(*iii*) $\frac{1}{3}(a+b+c \pm \sqrt{a^2+b^2+c^2-ab-bc-ca})$.

5. (*i*) $x=1, y=2.$ $x=-3, y=-2$;
(*ii*) $x=2, y=3$; $x=3, y=2$;
(*iii*) $x=2, y=3$; $x=3, y=2$;

Pages 52, 53, 54.

1. (*i*) n, (*ii*) $\frac{1}{4}(5-n)$. (*iii*) $5n$, (*iv*) $105-5n$, (*v*) $6n+1$, (*vi*) $44-11n$, (*vii*) $\frac{1}{3}(n-2)$. (*viii*) $1-4n$, (*ix*) $a-nd$. (*x*) $a+2nd$.

2. (*i*) $\frac{n(n+1)}{2}$. (*ii*) $\frac{n}{8}(9-n)$ (*iii*) $\frac{5}{2}n(n+1)$

(*iv*) $\frac{n}{2}(205-5n)$, (*v*) $n(3n+4)$ (*vi*) $\frac{11n}{2}(7-n)$, (*vii*) $\frac{1}{6}n(n-3)$

(*viii*) $-n(2n+1)$, (*ix*) $\frac{n}{2}(2a-\overline{n+1}d$, (*x*) $n(a+nd)$.

3. (*i*) 125th, (*ii*) 21st, (*iii*) 101st.
4. (*i*) 20, (*ii*) 1, 40. (*iii*) impossible.
5. (*i*) 101, (*ii*) 53, (*iii*) 200.
6. (*i*) 3, 5, 7,...(*ii*) 12, 23, 34,...(*iii*) 21, 22, 23,...
7. (*i*) 1, 2, 3,...(*ii*) $a+b$, $3a+b$, $5a+b$, $7a+b$,...
   (*iii*) 0, 2, 4, 6, 8,...
8. 35700. 9. 40992. 10. 35133. 11. 1, 5, 9,...
12. $n=10$. 13. (*i*) $\frac{1}{2}(x^3-y^2+3x-y)$, (*ii*) $1+\frac{1}{2}n(n-1)$.
   (*iii*) $\frac{n-1}{2}$. 14. 200,000. 18. 21 [Read $\frac{-1}{21}$ insted on $\frac{-1}{20}$].

22. 99 : 119. (*ii*) $19a+b : 19c+d$.

Page 57.

1. (*i*) $27\frac{1}{2}$, (*ii*) 0, (*iii*) a^2+b^2.
2. (*i*) $7+d$, $7+2d$,..., where $d=\frac{125}{51}$.
   (*ii*) $5\frac{2}{7}$, $5\frac{3}{7}$, $5\frac{4}{7}$, $5\frac{5}{7}$, $5\frac{6}{7}$, 6, $6\frac{1}{7}$.
   (*iii*) 1, $1+d$, $1+2d$,..., where $d=n+1$.
   (*iv*) a, $a+d$, $a+2d$,..., where $d=\frac{n^3}{n+1}$.
3. $n=0$.

Pages 58, 59, 60.

8. $d=2$. 12. 2, 8, 14. 13. 1, 3, 5.
14. 3, 5, 7, 9, ; $6+3\sqrt{39}$, $6+\sqrt{39}$, $6-\sqrt{39}$, $6-3\sqrt{39}$.
15. 3, 5, 7, 9. 16. 2, 5, 8. 17. 1, 4, 7. 18. $\frac{2}{15}$, $\frac{5}{15}$, $\frac{8}{15}$
19. Rs. 20. Rs. 380. 20. 8. (Read 100^2 instead of 20^2).

Pages 64, 65

1. (*i*) 3^{n-1}, $\frac{1}{2}(3^n-1)$; (*ii*) $(-1)^{n-1}2^n.9^{1-n}$, $\frac{9}{5}[1-(-\frac{2}{3})^n]$

(*iii*) $(\sqrt{2}-1)^{n-2}$, $2+\frac{3}{2}\sqrt{2}-2\sqrt{3}.(\sqrt{2}-1)^{n-2}$.

(*iv*) $(a-1)(\sqrt{a}-1)^{n-2}$, $\dfrac{(\sqrt{a}-1\,[1-(\sqrt{a}+1)^n]}{2-\sqrt{a}}$

[Read the second and the third term as $a-1$, $(a-1)(\sqrt{a}-1)$

2. $a^{\frac{n-q}{p-q}}$, $b^{\frac{p-n}{p-q}}$ 7. 2, 4, 8, 16.

8. (*i*) $\dfrac{1}{x-y}\left(\dfrac{x^2(1-x^n)}{1-x}-\dfrac{y^2(1-y^n)}{1-y}\right)$.

(*ii*) $\dfrac{x^2(1-x^{2n})}{1-x^2}+\dfrac{xy(1-x^ny^n)}{1-xy}$.

(*iii*) $\dfrac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)}$.

(*iv*) $a.\dfrac{a^n-1}{a-1}+\frac{1}{2}bn(n+1)$

(*v*) $\frac{1}{2}n(n+5)x+\dfrac{x^2}{1+x}[1-(-x)^n]$

(*vi*) $2(n-1+2^{-n})$. (*vii*) $1+\dfrac{b(1+c)(1-b^{n-1}c^{n-1})}{1-bc}$

(After the first term 1, put every two terms in a bracket.)
(*viii*) $2^{n+1}-2^{-n}$.

9. (*i*) $\dfrac{10^{n+1}}{27}-\dfrac{10}{27}-\dfrac{n}{3}$; (*ii*) $\dfrac{4}{81}.10^{n+1}-\dfrac{40}{81}-\dfrac{4}{9}n$.

(*iii*) $\dfrac{1}{3}n-\dfrac{1}{27}\left(1-\dfrac{1}{10^n}\right)$. (*iv*) $\dfrac{7}{9}n-\dfrac{7}{81}\left[1-\dfrac{1}{10^n}\right]$.

(*v*) $\dfrac{5}{8}\left[1-\dfrac{1}{3^n}\right]$. (*vi*) $\dfrac{41}{333}\left[1-\dfrac{1}{10^{3n}}\right]$.

10. (*i*) Sum to infinity$=\frac{5}{8}$, (*ii*) Sum to infinity$=\frac{123}{220}$.

Pages 68, 69, 70.

1. 2 ; 2. $\frac{3}{4}$; 3. $\frac{7}{24}$; 4. $\frac{25}{114}$; 5. $\dfrac{1}{1-x}$.

6. $\dfrac{3\sqrt{3}}{2}$; 7. $\dfrac{10}{3}$; 8. $\dfrac{250}{9}$; 9. $\dfrac{25}{14}$.

10. $\frac{1}{1-r} \cdot \frac{1}{1-br}$. **11.** $\frac{5}{9}$ **12.** $\frac{17}{45}$. **13.** $\frac{23183}{9900}$.

14. $\frac{119}{468}$. **15.** $\frac{8}{7}-\frac{8}{49}+\frac{8}{343}-\ldots\ldots\ldots$

16. $\frac{3}{4}, \frac{1}{4}, \frac{1}{12}, \ldots\ldots\ldots$ **17.** $12, 3, \frac{3}{4}, \ldots\ldots$

18. (*i*) 15, 45, 135, 405 ; (*ii*) $-28, 14, -7, \frac{7}{2}, \frac{-7}{2}, \frac{7}{8}$.

21. 1, 49. **22.** 4, 16. **23.** $n=-\frac{1}{2}$.

24. $(ab)^{\frac{1}{n+1}} \cdot \dfrac{b^{\frac{n}{n+1}}-a^{\frac{n}{n+1}}}{b^{\frac{1}{n+1}}-a^{\frac{1}{n+1}}}$.

Page 71.

1. 1, 2. **2.** 1, 2, 4, 8. **3.** Read $x^b\ y^c\ z^a=x^c\ y^a\ z^b$
4. 4, 8, 16, 32. **5.** $\frac{2}{3}, -1, \frac{3}{2}$.

Pages 77, 78, 79.

1. $\frac{60}{16-n}$. **2.** $\frac{35}{9-2n}$. **3.** $\frac{8}{n}$. **4.** $\frac{30}{53-13n}$ (Read the series as $\frac{3}{4}+\frac{10}{6}+\frac{15}{7}+\ldots\ldots$). **5.** $\frac{1}{3}, \frac{3}{10}, \frac{3}{11}, \frac{1}{4}, \frac{3}{13}$.
6. $\frac{1}{3}, \frac{2}{7}, \frac{1}{4}, \frac{2}{9}, \frac{1}{5}$. **7.** $\frac{1}{11}$. **8.** $\frac{17}{16}, \frac{17}{15}, \frac{17}{14}; \ldots \frac{17}{3}$.
9. $\frac{57}{15}, \frac{57}{11}, \frac{57}{7}$. **10.** 2, 18. **11.** 29, 21. **12.** pq. **14.** $\sqrt{ab}$.

Pages 83, 84.

1. $\frac{1}{3}n(4n^2+18n+23)$. **2.** $\frac{1}{4}n(n+1)(n^2+13n+42)$.
3. $\frac{1}{12}n(n+1)(3n^2+11n+10)$ **4.** $\frac{1}{3}n(4n^2-1)$.
5 $n(3n^2+6n+1)$. **6.** $\frac{1}{12}n(6n^2+15n+11)$.
7. $\frac{1}{2}n(6n^2-3n-1)$. **8.** $\frac{1}{12}n(n+1)^2(n+2)$.
9. $\frac{1}{6}n(n+1)(n+2)$. **10.** $3n^2-4n$.
12. $\frac{3}{2}n(n+1)+\frac{3}{2}(3^n-1)$.
13. $\pm\frac{1}{2}n(n+1)$ according as n is even or odd.
15. $\frac{1}{24}(n-1)n(n+1)(3n+2)$.

Pages 86, 87.

1. $\frac{1}{2}n(n+1)$, $\frac{1}{6}n(n+1)(n+2)$.
2. n^2+1, $\frac{1}{6}n(2n^2+3n+7)$.
3. $2n^2+n+1$, $\frac{1}{6}n(4n^2+9n+11)$.

4. $2^n+1,\ 2^{n+1}+n-2$.
5. $2^{n+1}-3,\ 2^{n+} -3^n-4$.
6. $\frac{1}{3}n(n^2+1),\ \frac{1}{8}n(n+1)(n^2+n+2)$.
7. $2^{2n-2}+2^{2n-3}-2^{n-2},\ 2^{2n-1}-2^{n-1}$.
8. $\frac{1}{(3n-1)(3n+2)},\ \frac{n}{2(3n+2)}$.
9. $\frac{1}{n(n+1)},\ \frac{n}{n+1}$. 10. $\frac{2}{n(n+1)},\ \frac{2n}{n+1}$.
11. $\frac{1}{6}$, 1, 2. 13. $\frac{3}{4}-\frac{1}{4}\cdot\frac{4n+3}{(2n+1)(n+1)}$.
14. $\frac{11}{18}-\frac{1}{9}\cdot\frac{27n^2+36n+11}{(n+1)(3n+1)(3n+2)}$.
15. $\frac{1}{4}(\sqrt{4n+1}-1)$, the series is divergent.

Page 90.

1. $\frac{1}{1+x}+\frac{x^2[1-(-x)^{n-2}]}{(1+x)^2}-\frac{(-1)^n(n-1)x^n}{1+x}$.
2. $\frac{1}{(1-x)^2}+2x\frac{1-x^{n-1}}{(1-x)^3}-(n-1)^2\frac{x^n}{(1-x)^2}+n^2\frac{x^{n+1}}{(1-x)^2}$.
3. $\frac{1-x^n}{(1-x)^3}-\frac{n(n+3)}{2}\cdot\frac{x^n}{(1-x)^2}+\frac{n(n+1)}{2}\cdot\frac{x^{n+1}}{(1-x)^3}$
4. $\frac{5}{6}-\frac{5}{18}\left[1-(-\frac{1}{5})^{n-1}\right]+\frac{(-1)^{n+1}(2n-1)}{6\cdot5^{n-1}}$.
5. $\frac{11}{6}+\frac{5}{48}\left(1-\frac{1}{5^{n-2}}\right)-\frac{n}{12\cdot5^{n-3}}$.
6. $\frac{1}{(1-x)^2}+2x\frac{1-x^{n-1}}{(1-x)^3}-(n-1)^2\cdot\frac{x^n}{(1-x)^2}+n^2,\ \frac{x^{n+1}}{(1-x)^2}$.
7. 3. 8. $\frac{2}{3}-\frac{4}{9}[1-(-\frac{1}{2})^{n-1}]-\frac{2}{3}(2n-1)(-\frac{1}{2})^n$; sum to infinity is $\frac{2}{9}$.
9. $\frac{1}{1-x}\left[\frac{1-r^nx^n}{1-rx}-\frac{1-r^n}{1-r}x^n\right],\ \frac{1}{(1-x)(1-rx)}$. 10. $\frac{64}{125}$.

Miscellaneous Exercises Page 90

1. $n^2(2n^2-1)$. 2. $\frac{n}{12}(3n^3+26n^2+81n+106)$.

3. $\frac{n}{4}(n^3+10n^2+35n+50)$. 4. $(n-1)2^n+1$.

5. **Hint**. First evaluate $\Sigma\, n^4$ by the method used for Σn^2 and Σn^3. Then proceed. The sum is found to be $\frac{1}{60}n(n+1)(3n^3+12n^2+13n+2)$.

6. $\frac{n(n+1)^2(n+2)}{12}$. 7. $\frac{27}{4}\left(1-\frac{1}{3^n}\right)-\frac{n}{2.3^{n-2}}$.

8. $2n-1+2^{n+1}-2^{-n}$. 9. $2n+\frac{a^n-b^n}{a-b}\left(\frac{a}{b^n}+\frac{b}{a^n}\right)$.

10. $n(n+1)^2$. 11. $\frac{n}{3}(5n^2-2)$. 12. $3\cdot 2^n-n-3$.

13. $\frac{19}{2}(3^n-1)-\frac{n}{2}$. 14. $\frac{17}{9}(4^n-1)-\frac{2}{3}n$.

15. $\frac{n}{a(a+2n)}$.

Revision Questions Pages 91, 92

2. $a^3x^3+abcx+c^3=0$. 4. $m=2, -\frac{10}{9}$.

5. (*i*) $\frac{ab+bc+ca\pm\sqrt{b^2c^2+c^2a^2+a^2b^2-abc(a+b+c)}}{a+b+c}$.

(*ii*) $-m, -n$; (*iii*) $1, -1\ \frac{1\pm\sqrt{13}}{2}$ (read the equation as $x^4-x^3-4x^2+x+3=0$). (*iv*) $4, -4$; (*v*) $x=\frac{1}{2}, y=0$.

8. $\frac{2ac-ab-bc}{2b-a-c}$. 10. (*i*) $\frac{n}{4}(27n^3+126n^2+153n+14)$.

(*ii*) $\frac{n(n+1)}{2}+1-2^{-n}$. (*iii*) $\frac{2}{3}$. (*iv*) $\frac{1}{6}n(n^3-2n^2+74n-67)$

(*v*) $\frac{20}{27}(10^n-1)-\frac{2}{3}n$; (*vi*) $\frac{2}{9}n-\frac{2}{81}\left(1-\frac{1}{10^n}\right)$.

(*vii*) $\frac{1}{2x}$.

Revision Exercises Pages 92, 93, 94.

I

6. (*i*) $\frac{n}{6n+9}$, sum to infinity $=\frac{1}{6}$; (*ii*) $\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}$, sum to infinity $=(1-x)^{-2}$.

II

1. $m=1. -\frac{1}{3}$. 2. $a^2c^2x^3+(2ac-b^2)x+1=0$.

6. $\frac{75}{32}$. 7. $\frac{1}{4}(n+1)^2$ or $-\frac{1}{4}n(n+2)$, according as n is odd or even.

III

1. (*i*) $\frac{94}{25}$; (*ii*) $3, -\frac{3}{2}$. 2. $\frac{60}{16-n}$.

5. $\frac{1+x+x^3}{1+2x+x^2}$. 6. 2, 4, 8.

IV

1. $-6\pm\sqrt{12}$. 6. 3, 7, 11 or 18, 7, -4.

7. $\frac{1}{4}n(n+1)(n+2)(n+3)$.

V

2. (*i*) $\frac{19\pm\sqrt{57}}{4}$; (*ii*) $x=9, y=3, x=9, y=-3$.

7. $\frac{n(n+1)}{2} \cdot x - \frac{n(n+1)(n-1)}{3(x-1)}$.

Pages 99, 100, 101.

1. 56. 2. 720. 3. 120. 4. (*i*) 24, (*ii*) 6, (*iii*) 6, (*iv*) 2. (*v*) 12. 5. 11520. 6. 120. 7. (*i*) 720, (*ii*) 120, (*iii*) 24, (*iv*) 240. (*v*) 480, (*vi*) 96, (*vii*) 192 (*viii*) 48, (*ix*) 288, (*x*) 240, (*xi*) 480. 8. (*i*) 24, (*ii*) 12, (*iii*) 12, (*iv*) 6, (*v*) 6, (*vi*) 6, (*vii*) 12, (*viii*) 4, (*ix*) 4, (*x*) 16. 9. (*i*) 300, (*ii*) 108, (*iii*) 60, (*iv*) 24, (*v*) 21. (*vi*) 12,

10. (*i*) 720, (*ii*) 120, (*iii*) 24, (*iv*) 240, (*v*) 480.

11. 8 !. 12. 96. 13. 1728. 14. 11880.

15. 11880. 16 30. 17. 63. 18. (*i*) 120, (*ii*) 5^5.

19. (*i*) 256, (*ii*) 252. 20. 60 or 10 according as the order in which he attempts the questions does or does not matter.

Pages 105, 106, 107.

1. (*i*) 8, (*ii*) 7, (*iis*) 6. 2. (*i*) 36, (*ii*) 8, (*iii*) 6, (*iv*) 5.

3. (*i*) 15, (Read 28 instead of 8). (*ii*) 18, (*iii*) 16, (*iv*) 6, (Read 12 instead of 2).

4. (*i*) $2n-3r+2=0$; (*ii*) $n-2r+1=0$, (*iii*) $n-r+1=0$, (*iv*) $2n=3r$.

15. (*i*) 8, (*ii*) 6. (*iii*) 9. 16 (*i*) 12, (*ii*) $r=8$, (read 16 for 15). 17. 18 19 !). 18. 6(7 !). 19. 600, 120. 20. 3 ! 4 !.

21. $m\,!\,.\,{}^{m+1}P_n$. 22. 93. 23. 96. 24. (*i*) 4.5.6.7.8. (*ii*) 81. 25. ${}^{n-1}P_r$.

Pages 109, 110.

1. (*i*) $\dfrac{9\,!}{2\,!\,4\,!}$, (*ii*) $\dfrac{8\,!}{2\,!\,3\,!}$, (*iii*) $\dfrac{7\,!}{2\,!\,3\,!}$, (*iv*) $\dfrac{8\,!}{4\,!}$, (*v*) $\dfrac{7\,.\,8\,!}{2\,.\,4\,!}$.

2. (*i*) $\dfrac{8\,!}{2\,!\,3\,!}$, (*ii*) $\dfrac{6\,!}{2\,!}$ (*iii*) $\dfrac{4\,!}{2\,!}$ (*iv*) $\dfrac{6\,!}{2\,!}$, (*v*) 1800.

3. (*i*) $\dfrac{13\,!}{2\,!\,2\,!\,3\,!\,4\,!}$, (*ii*) $\dfrac{8\,!}{(2\,!)^3}$, (*iii*) $\dfrac{11\,!}{4}$, (*iv*) $\dfrac{9\,!}{4}$, (*v*) 7 !, (*vi*) $\dfrac{6\,!}{4}$ (*i*) 10. (*ii*) 4, (*iii*) 3.

5. (*i*) 9, (*ii*) 9, (*iii*) 6. 6. 60. 7. $\dfrac{5(7\,!)}{4\,!}$.

Pages 120, 121, 122, 123.

1. (*i*) 210, (*ii*) 371. 2. 3960. 3. 35. 4. 495.
5, 816000. 6. 5775. 7. 816. 8. 1225. 9. 56, 21.
10. 6(7 !). 11. (*i*) ${}^nC_2-{}^pC_2+1$ (*ii*) ${}^nC_3-{}^pC_3$.
13. $\frac{1}{2}n(n-3)$. 14. (*i*) 2(6 !), (*ii*) 4(6 !). 15. 6 ! 7 !.
16. 10. 17. $\dfrac{(m+n)\,!}{m\,!\,n\,!}$. 18. 2856. 19. 360.

20. ${}^8C_4$ or $2.{}^5C_4$ according as the two beats are or are not alike. 21. 42. 22. $n=15$, $r=5$. (Read $r+1$ for $r+2$).

23. (*i*) $\dfrac{n\,!}{(n-m)\,!}$, (*ii*) m^n. 24. 344. 25. 2100, 756.
28. 210. 27. 5.8.13. 14−1. 23. $\frac{1}{2}$. ${}^{22}C_{11}$. 29. ${}^{27}C_{11}$.
31. $n=10$, $r=5$ (read ${}^{n+1}C_{r+1}$ in the question). 32. 32.
33. 256. 34. 72. 35. 9. 36. $\frac{1}{4}$. (8 !). 37. $\frac{1}{8}$(11 !). 3(8 !)
38. 2.(9 !). 39. $\frac{1}{2}(n-1)$! 40. $\dfrac{13\,!}{40}$. 41. 5 or 6. 43. 13.
44. 12. 45. 371.

Pages 129, 180.

1. (*i*) $x^{\frac{5}{2}}+\frac{5}{2}x^{\frac{3}{2}}+\frac{5}{2}x^{\frac{1}{2}}+\frac{5}{4}x^{-\frac{1}{2}}+\frac{5}{16}x^{-\frac{3}{2}}+\frac{1}{32}x^{-\frac{5}{2}}$,

(ii) $x^3-3x^2+4x-\frac{28}{9}+\frac{14}{9x}-\frac{14}{27x^2}+\frac{28}{243x^3}-\frac{4}{243x^4}+\frac{1}{729x^5}$
$-\frac{1}{19683x^6}$.

(iii) $1+8x^{-\frac{1}{2}}+28^{-1}+56x^{-\frac{3}{2}}+70x^{-2}+56x^{-\frac{5}{2}}$
$+28x^{-2}+8x^{-\frac{7}{2}}+x^{-4}$.

(iv) $x^6y^{-6}-6x^4y^{-4}+15x^2y^{-2}+20+15x^{-2}y^2$
$-6x^{-4}y^4+x^{-6}y^6$.

2. (i) $1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$.
(ii) $1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-5x^7+x^8$.

3. (i) 1004006004001, (ii) 992023968016, (iii) 996005996001.

4. $2(x^5+10x^3y^2+5xy^4)$, (ii) $2(6x^5y+20x^3y^3+6xy^5)$.
(iii) $2\sqrt{x^2-y^2}(64x^6-80x^4y^2+24x^2y^4-y^6)$.

5. (i) ${}^{10}C_r.(-1)^r.2^{10-r}.3\ .\ x^{20-3r}$.
(ii) ${}^{25}C_r.\ 5^{25-r}.8^r.a^{50-2r}.x^r$.
(iii) ${}^{n}C_r.a^{n-r}.b^r.x^{pn-p-r-qr}$.
(iv) ${}^{4n}C_r.x^{4n-4r}$. (v) ${}^{9}C_r.2^{9-r}.3^{-r}.x^{9-3r}$.
(vi) ${}^{8n}C_r\,(-1)^r.x^{8n-2r}$, (vii) ${}^{3n}C_r.x^{6n-4r}$.
$(viii)$ ${}^{12}C_r.(-1)^r.2^{12-r}.x^{24-3r}$, (ix) ${}^{2n+1}C_r\,x^{2n-2r+1}.y^{3r-2n-1}$.
(x) ${}^{10}C_r.x^{10-2r}$.

6. (i) -1959552, (ii) 13440, (iii) ${}^{25}C_{11}.5^{14}.2^{33}.a^{28}$.

9. ${}^{3n}C_n.x^{-n}$. 10. $n=8$. 11. (i) 7920, (ii) $\frac{7}{9}.2^8$.

(iii) $\frac{(4p)\,!}{(2p)\,!\,(2p)\,!}$, (iv) $(-1)^{\frac{4n}{3}}.\ {}^{4n}C_{\frac{4n}{3}}$, (v) ${}^{3n}C_{\frac{3n}{2}}$,

(vi) $\frac{7}{18}$.

12. (i) ${}^{12}C_5$, (ii) 35, (iii) $\frac{(m+n)\,!}{m\,!\,n\,!}$.

13 (i) $\frac{20\,!}{(10\,!)^2}x^{10}$, (ii) ${}^{11}C_5.a^{12}.x^{10}$, ${}^{11}C_6.a^{10}.x^{12}$,

(iii) $\frac{(2n)\,!}{(n\,!)^2}$, (iv) $14.3^7.a^{17}.b^{-4}, -14.3^6.a^{16}.b^{-5}$.

(v) $-\frac{10\,!}{(5\,!)^2}$, (vi) $(-1)^p.\frac{(2p+1)\,!}{p\,!\,(p+1)\,!}a^{p+1}.b^p.x^{1-p}$,

$(-1)^{p+1}\dfrac{(2p+1)\,!}{p\,!\,(p+1)\,!}\cdot a^p.b^{p+1}\,x^{-p-2},$

(*vii*) $\dfrac{100\,!}{(50\,!)^2}x^{50}.$

Pages 132, 133.

1. 2^{25}. 2. $2^{12}-1$. 3. $2^{14}-1$. 4. $2^{16}-2$.
5. $2^{20}-42$.

Pages 135.

1. 5th term, *i. e.*, $2^6.{}^{10}C_4.5^4.3^{-4}$.
2. 7th term. 3. (*i*) 6th term, (*ii*) 8th term, (*iii*) 1st term. 5. $(2+3)^6$.

Pages 142, 143.

1. (*i*) $\dfrac{3.5.7\ldots(2r+1)}{r\,!}\,x^r.$

(*ii*) $(-1)^r\,\dfrac{p(p+q)(p+2q)\ldots(p+\overline{r-1}q)}{r\,!\,q^r}\,x^r.$

(*iii*) $\dfrac{1.5.9\ldots(4r-3)}{r\,!\,4^r}\,x^r.$

(*iv*) $\dfrac{(p+1)(2p+1)\ldots[(r-1)p+1]}{r\,!}\,x^r.$

(*v*) $\dfrac{3.7.11\ldots(4r-1)}{r\,!\,4^r}\,x^r$; (*vi*) $\dfrac{n(n+1)\ldots(n+r-1)}{r\,!}\,x^r.$

2. (*i*) $\dfrac{x^2}{a^8}\left(1+\dfrac{4x}{a^{\frac{5}{3}}}+\dfrac{10x^2}{a^{\frac{16}{3}}}+\dfrac{20x^3}{a^8}\right).$

(*ii*) $a-3a^{\frac{4}{3}}.b^{-\frac{1}{3}}+6a^{\frac{5}{3}}.b^{-\frac{2}{3}}-10a^2.b^{-1}+\ldots\ldots$

(*iii*) $\dfrac{a^4}{81}+\dfrac{8}{243}a^5x+\dfrac{40}{729}a^6x^2+\dfrac{160}{2187}a^7x^3+\ldots\ldots$

(*iv*) $a^{-\frac{p}{q}}\left(1-\dfrac{pb}{qa}x+\dfrac{p(p+q)}{1.2}\cdot\dfrac{b^2x^2}{a^2q^2}\right.$

$\left.-\dfrac{p(p+q)(p+2q)}{1.2.3}\cdot\dfrac{b^3x^3}{a^3q^3}+\ldots\ldots\right).$

(v) $2^{-\frac{4}{5}}\left(1+\frac{6}{5}x^2+\frac{81}{50}x^4+\frac{567}{250}x^6+\ldots\ldots\right)$

(vi) $2^{44}+11.2^{41}.x+99.2^{37}.x^2+231\,2^{34}.x^3+\ldots\ldots$

3. $2p^3+2p+1$. 4. 3. 5. $1+2x-\frac{3}{2}x^2+3x^3+\ldots\ldots$

6. 21504. 8. (i) 3rd, (ii) 6th, (iii) 8th, (iv) 3rd, (v) 68th, (vi) 5th, (vii) $(p+3)$rd.

9. $\dfrac{(2r)!}{(r!)^2}$. 12. 3. 13. $\frac{65}{16}$, the third term.

Pages 151, 152, 153.

1. 9·9499. 2. 10·0999. 3. 3·0024.
4. 1·00066. 5. 1·0332. 6. 1·987.
7. 1·1705. 8. (i) ·91917, (ii) 3·2162.

(iii) 2.002. 15. 3. 24. (i) $(1-\frac{3}{4})^{-\frac{5}{2}}=32$,

(ii) $(1+\frac{1}{2})^{-\frac{1}{2}}=\sqrt{\frac{2}{3}}$, (iii) $(1-\frac{1}{9})^{-\frac{7}{2}}$, (iv) $(1-\frac{1}{2})^{-\frac{8}{3}}=2^{\frac{8}{3}}$

Page 157.

1. $\dfrac{2}{x+3}-\dfrac{1}{x+2}$. 2. $\dfrac{1}{a-b}\left(\dfrac{a}{x-a}-\dfrac{b}{x-b}\right)$.

3. $\dfrac{1}{x+1}+\dfrac{1}{x-1}-\dfrac{1}{x}$.

4. $-\dfrac{1}{6}\cdot\dfrac{1}{x+1}+\dfrac{4}{15}\cdot\dfrac{1}{x-2}+\dfrac{9}{10}\cdot\dfrac{1}{x+3}$.

5. $x+1+\dfrac{5}{2x+3}+\dfrac{2}{x-4}$.

6. $1+\dfrac{(c-a)(c-b)}{c-d}\cdot\dfrac{1}{x-c}+\dfrac{(d-a)(d-b)}{d-c}\cdot\dfrac{1}{x-d}$.

7. $\dfrac{1}{a+b}\left(\dfrac{pa+q}{x-a}+\dfrac{pb-q}{x+b}\right)$.

8. $2+\dfrac{2}{a-b}\left(\dfrac{a^2}{x-a}-\dfrac{b^2}{x-b}\right)$.

9. $x+6+\frac{1}{2}\cdot\dfrac{1}{x-1}-\dfrac{16}{x-2}+\dfrac{81}{2}\cdot\dfrac{1}{x-3}$

10. $\dfrac{2}{x-1}+\dfrac{3}{x-2}-\dfrac{4}{x-3}$.

11. $-\frac{1}{2}\cdot\frac{1}{x-1}+\frac{1}{2}\cdot\frac{1}{x-3}$

12. $1-\frac{1}{x}-\frac{2}{x+1}+\frac{4}{2x+1}$.

13. $\frac{a^2(b-c)}{1-ax}+\frac{b^2(c-a)}{1-bx}+\frac{c^2(a-b)}{1-cx}$.

14. $\frac{3}{5}\cdot\frac{1}{x-1}-\frac{1}{3}\cdot\frac{1}{x-2}-\frac{4}{15}\cdot\frac{1}{x+4}$.

15. $\frac{a^2+pa+q}{(a-b)(a-c)}\cdot\frac{1}{x-a}+\frac{b^2+pb+q}{(b-a)(b-c)}\cdot\frac{1}{x-b}$

$+\frac{c^2+pc+q}{(c-a)(c-b)}\cdot\frac{1}{x-c}$.

16. $\frac{2}{x^2+4}-\frac{1}{x^2+3}$.

17. $1+\frac{(a^2-c^2)(b^2-c^2)}{d^2-c^2}\cdot\frac{1}{x^2+c^2}+\frac{(a^2-d^2)(b^2-d^2)}{c^2-d^2}\cdot\frac{1}{x^2+d^2}$.

18. (i) $\frac{f(a)}{(a-b)(a-c)}\cdot\frac{1}{x-a}+\frac{f(b)}{(b-a)(b-c)}\cdot\frac{1}{x-b}$

$+\frac{f(c)}{(c-a)(c-b)}\cdot\frac{1}{x-c}$.

(ii) $R+\frac{f(a)}{(a-b)(a-c)}\cdot\frac{1}{x-a}+\frac{f(b)}{(b-a)(c-b)}\cdot\frac{1}{x-b}$

$+\frac{f(c)}{(c-a)(c-b)}\cdot\frac{1}{x-c}$.

Page 159.

1. $\frac{2x-11}{x^2+11}-\frac{2}{x-4}$.

2. $\frac{5x-3}{2(x^2+2x+3)}-\frac{1}{2}\cdot\frac{1}{x-3}$.

3. $\frac{3}{5}\cdot\frac{1}{x-1}-\frac{1}{4}\cdot\frac{1}{x-2}-\frac{7x-2}{20(x^2+4)}$.

4. $1-\frac{4}{x-1}+\frac{4}{x+1}+\frac{13}{2\sqrt{2}}\left(\frac{1}{x-\sqrt{2}}-\frac{1}{x+\sqrt{2}}\right)$.

5. $\frac{1}{2}\cdot\frac{1}{x-1}+\frac{1}{2}\cdot\frac{1}{x+1}+\frac{x}{x^2+1}$.

6. $\dfrac{2}{x-1}-\dfrac{x+5}{x^2+2x+5}$.

7. $x+\dfrac{1}{3}\cdot\dfrac{1}{x-1}-\dfrac{1}{3}\cdot\dfrac{x-1}{x^2+x+1}$.

8. $\dfrac{1}{3}\cdot\dfrac{1}{x+1}-\dfrac{1}{3}\cdot\dfrac{x-2}{x^2-x+1}$.

9. $\dfrac{1}{5}\cdot\dfrac{x+12}{x^2+1}-\dfrac{1}{5}\cdot\dfrac{1}{x+3}$.

10. $\dfrac{1}{2\sqrt{2}}\left(\dfrac{x+\sqrt{2}}{x^2+x\sqrt{2}+1}-\dfrac{x-\sqrt{2}}{x^2-x\sqrt{2}+1}\right)$.

Page 162, 163.

1. $\dfrac{1}{2}\cdot\dfrac{1}{x+1}+\dfrac{1}{2}\cdot\dfrac{1}{(x+1)^2}-\dfrac{1}{2}\cdot\dfrac{x}{x^2+1}$.

2. $\dfrac{7}{5}\cdot\dfrac{1}{(x-3)^2}+\dfrac{3}{25}\cdot\dfrac{1}{x-3}-\dfrac{3}{25}\cdot\dfrac{1}{x+2}$.

3. $\dfrac{3}{2}\cdot\dfrac{1}{(x-1)^3}+\dfrac{3}{4}\cdot\dfrac{1}{(x-1)^2}+\dfrac{1}{8}\cdot\dfrac{1}{x-1}-\dfrac{1}{8}\cdot\dfrac{1}{x+1}$.

4. $\dfrac{2}{(x-2)^3}-\dfrac{1}{3}\cdot\dfrac{1}{(x-2)^2}+\dfrac{1}{9}\cdot\dfrac{1}{x-2}-\dfrac{1}{9}\cdot\dfrac{1}{x+1}$.

5. $\dfrac{1}{x^4}-\dfrac{1}{x^3}+\dfrac{1}{x^2}-\dfrac{1}{x}+\dfrac{1}{x+1}$.

6. $\dfrac{1}{x-a}+\dfrac{1}{2}\cdot\dfrac{a}{(x-a)^2}+\dfrac{1}{2}\cdot\dfrac{1}{x^2+a^2}$.

7. $\dfrac{1}{9}\cdot\dfrac{1}{x+2}-\dfrac{1}{3}\cdot\dfrac{1}{(x+2)^2}-\dfrac{1}{9}\cdot\dfrac{1}{x-1}+\dfrac{2}{3}\cdot\dfrac{1}{(x-1)^2}$.

8. $\dfrac{2}{1-x}+\dfrac{1}{(1-x)^2}-\dfrac{5}{(x-2)^2}+\dfrac{2}{x-2}$.

9. $\dfrac{3}{2}\cdot\dfrac{1}{(x-1)^3}+\dfrac{3}{4}\cdot\dfrac{1}{(x-1)^2}+\dfrac{1}{8}\cdot\dfrac{1}{x-1}-\dfrac{1}{8}\cdot\dfrac{1}{x+1}$.

10. $\dfrac{5}{(x-1)^4}-\dfrac{7}{(x-1)^3}+\dfrac{1}{(x-1)^2}+\dfrac{3}{x-1}$.

Page 162.

1. $\frac{1}{4}\cdot\frac{1}{1-x}+\frac{1}{4}\cdot\frac{x+1}{x^2+1}-\frac{1}{2}\frac{x+1}{(x^2+1)^2}$.

2. $\frac{5}{4}\cdot\frac{1}{1+x}+\frac{5}{4}\cdot\frac{x+1}{x^2+1}+\frac{5}{2}\cdot\frac{x-1}{(x^2+1)^2}$.

3. $\frac{7}{36}\cdot\frac{1}{x+2}-\frac{7}{36}\cdot\frac{x-2}{x^2+2}-\frac{7}{6}\cdot\frac{x-2}{(x^2+2)^2}$.

4. $\frac{3}{4}\cdot\frac{1}{x+1}+\frac{1}{4}\cdot\frac{1}{(x+1)^2}-\frac{1}{4}\cdot\frac{3x-2}{x^2+1}-\frac{1}{2}\cdot\frac{x-1}{(x^2+3)^2}$.

5. $\frac{1}{9}\cdot\frac{1}{x}+\frac{2}{9}\cdot\frac{1}{x^2}-\frac{1}{9}\cdot\frac{x+2}{x^2+3}-\frac{1}{3}\cdot\frac{x-1}{(x^2+3)^2}$.

Pages 162, 163.

6. $\frac{n}{6n+4}, \frac{1}{6}$. 7. $\frac{1}{2}\left(\frac{1}{2}-\frac{1}{n+1}+\frac{1}{n+2}\right), \frac{1}{4}$.

8. $\frac{1-r^n}{1-r}\cdot\frac{1}{(1+x)(1+r^nx)}, \frac{1}{x(x+1)(r-1)}$.

9. $\frac{6n^2+5n}{8n^2+12n+4}, \frac{3}{2}$. 10. $\frac{1}{x}\left(\frac{1}{1+x}-\frac{1}{1+2^n.x}\right)$.

Punjab University Papers

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1. $px^2+(q+p^2)x+pq=0$; $x=4, y=9$; $x=9, y=4$.
$x=-3+4\sqrt{-10}, y=-3-4\sqrt{-10}$;
$x=-3-4\sqrt{-10}, y=-3+4\sqrt{-10}$

2. (i) $\frac{a}{1-r}+\frac{br}{(1-r)^2}$; (ii) $\frac{n(n+1)(n+2)}{6}$.

3. $r(r-1)(r-2).\frac{(n-3)!}{(n-r)!}$; 35, considering the departments to be all alike.

4. $\frac{(3n)!}{n!\,(2n)!}.x^n$; 1·25 ; $-\frac{1}{8}\cdot\frac{1}{x+1}+\frac{1}{8}\cdot\frac{1}{x-1}$
$+\frac{3}{4}\cdot\frac{1}{(x-1)^2}+\frac{3}{2}\cdot\frac{1}{(x-1)^3}$.

1940

1. $p^3+q(1+q)=3pq$; $2x^2-9q=0$.
$x=2, y=3$ and $x=\frac{5}{2}, y=\frac{5}{2}$.
2. 4, 8, 16, 32 ; $\frac{1}{6}n(n+1).(n+2)$.
3. 2.91
4. $\frac{7}{18}, n+1, \frac{x+12}{5(x^2+1)}-\frac{1}{5}\cdot\frac{1}{x+3}$.

1941

1. (b) (1) $x=2, y=3$; $x=-2, y=-3$; $x=-3, y=-2$; $x=3, y=2$; (2) 0, 3.
2. (a) $\frac{44}{45}$; (b) $3n^2-4n$.
3. $\frac{11\,!}{8}$, 840.
4. 5376, 1·00067.

1942

1. (c) $y^2-2py+4q=0$; (d) $x=a, y=b$; $x=y=\frac{a+b}{2}$.
3. (b) $2n^2+n+1$, $\frac{20}{5n+3}$; (c) $\frac{1}{12}(n+1)^2(n+2)$.
4. (a) $\frac{13\,!}{40}$; (c) 144.
5. $\frac{1}{2}\cdot\frac{1}{1-x}-\frac{4}{1-2x}+\frac{9}{2(1-3x)}$.

1943

1. (b) (1) real, rational, unequal and both positive ;
(2) irrational, unequal and of opposite signs.
(c) $m=3, -2$; (d) $x^2+4x-1=0$.
2. (b) $p=5$; (c) 3, 4, 5, 6, 7.
3. (b) $(1+x)(1-x)^{-3}$.
4. (a) ${}^{n-4}P_r$; (b) 600, 120.
5. (b) $\frac{1}{a^3(x-a)}-\frac{2}{a^3x}-\frac{1}{a^3x^2}+\frac{x}{a^3(x^2+a^2)}$.

1944

1. (b) $x=\frac{1}{8}, y=\frac{4}{8}$ and $x=3, y=2$.
(c) $m=\frac{1}{8}, m\to\infty$.

2. (a) $\frac{n}{2}(2x+\overline{n-1}y)$.

(c) 9, 6, 4 ; 469.

3. (b) $n^2(2n^2-1)$. 4. (b) 40320 ; 720.

(c) $1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$.

5. (a) 5 01330,

(b) (i) $\frac{1}{x+1}+\frac{1}{3}\cdot\frac{x+1}{x^2-x+1}$.

(ii) $\frac{1}{2}\cdot\frac{1}{1+x}+\frac{1}{2}\cdot\frac{1}{(1+x)^2}-\frac{1}{2}\cdot\frac{x}{x^2+1}$.

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INTERMEDIATE
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